

## COMMUTING INVOLUTION GRAPHS OF LINEAR GROUPS

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ABSTRACT. In this paper, we determine the diameter of the commuting involution graphs of special and general linear groups over an arbitrary field. It turns out that our results also determine the diameter for certain projective special linear groups over finite fields. Moreover, we find the diameter of the commuting graphs of general linear groups on the set of all involutions over a field of characteristic 2, which completes the diameter of general linear groups on the set of all involutions. As an application, we classify the structure of the four-dimensional linear groups over finite fields according to the distance from a fixed involution.

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## 1. INTRODUCTION

Let  $X$  be a subset of a group  $G$ . The *commuting graph* of  $G$  on  $X$ , denoted by  $\Gamma(G, X)$ , is the graph with the vertex set  $X$  in which distinct elements  $x, y \in X$  are connected by an edge if  $xy = yx$ . In particular, if  $X$  is a conjugacy class of involutions, the corresponding graph is called the *commuting involution graph*. The *distance*  $d(x, y)$  between  $x$  and  $y$  is the number of edges in a shortest path connecting them. The *diameter* of a connected graph  $\Gamma(G, X)$ , denoted by  $\text{Diam}(\Gamma(G, X))$ , is the maximum distance over all pairs of vertices.

The commuting graph on the set of involutions was first studied by Brauer and Fowler in [3], where they show that the commuting graph of a finite group is connected with diameter at most 3 if it has more than one conjugacy class. This result is one of the key ingredients in the recent work of Morgan and Parker [9] on an upper bound of  $\text{Diam}(\Gamma(G, X))$  of a finite centerless group  $G$  on  $X = G \setminus \{1\}$ . Indeed, this result can be viewed as a generalization of Segev and Seitz's result [2] on finite simple classical groups, which in turn have played a major role in their proof of the Margulis-Platanov conjecture for inner forms of anisotropic groups of type  $A_n$ .

The concept of the commuting involution graph was introduced by Fischer [10], where he investigated three-transposition groups generated by conjugacy classes of involutions, which led eventually to the discovery of three new sporadic groups. An intensive study of commuting involution graphs was conducted by Bates, Bundy, Perkins, and Rowley. For instance, the commuting involution graphs for finite Coxeter groups, symmetric groups, and special linear groups have been investigated in [4, 5, 6]. See also more recent work on three-dimensional unitary groups [8].

The main theme of the present paper is to explore the commuting involution graphs of general and special linear groups over an arbitrary field. As mentioned above, this work was initiated in [6], and in particular [6] provides an upper bound for the diameter of the commuting involution graphs for special linear groups

over a field of characteristic 2 [6, Theorem 1.3]. The present paper establishes the diameter over an arbitrary field by improving this upper bound.

Let  $x$  be an involution (i.e., an element of order 2) in  $\mathrm{GL}_n(F)$ . Then, the element  $x$  is similar to

$$I(n, k) := \begin{cases} \left( \begin{array}{c|c} I_{n-k} & \\ \hline & -I_k \end{array} \right) & \text{if } \mathrm{char}(F) \neq 2, \\ \left( \begin{array}{c|c} I_{n-k} & \\ \hline & I_k \end{array} \right) & \text{if } \mathrm{char}(F) = 2, \end{cases}$$

for some  $1 \leq k \leq n$  in the first case and  $1 \leq k \leq n/2$  in the second case. The conjugacy class of  $I(n, k)$  will be denoted by  $X_k$ . Since  $X_n$  consists of a single element  $-I_n$  if  $\mathrm{char}(F) \neq 2$ , we will not consider this trivial case.

In this paper, we completely determine the diameter of commuting involution graph  $\Gamma(G, X_k)$  for  $G = \mathrm{GL}_n(F)$  and  $\mathrm{SL}_n(F)$  over an arbitrary field  $F$  as follows:

**Theorem 1.1.** *Let  $G = \mathrm{GL}_n(F)$  over an arbitrary field  $F$  and let  $X_k$  be the class of involutions for  $1 \leq k \leq n-1$  in case  $\mathrm{char}(F) \neq 2$  and  $1 \leq k \leq n/2$  in case  $\mathrm{char}(F) = 2$ . Then, for any  $n \geq 3$  and any  $X_k$ , the commuting involution graph  $\Gamma(G, X_k)$  is connected of diameter at most 4. More precisely,*

- (i) *If  $4k \leq n$  or  $4(n-k) \leq n$ , then  $\mathrm{Diam} \Gamma(G, X_k) = 2$ .*
- (ii) *If  $2k < n < 4k$  or  $2(n-k) < n < 4(n-k)$  or  $n = 2k$  with even  $k$ , then  $\mathrm{Diam} \Gamma(G, X_k) = 3$ .*
- (iii) *If  $n = 2k$  with odd  $k$ , then  $\mathrm{Diam} \Gamma(G, X_k) = 3$  for  $\mathrm{char}(F) \neq 2$  and  $\mathrm{Diam} \Gamma(G, X_k) = 4$  for  $\mathrm{char}(F) = 2$ .*

We would like to point out that our diameter for  $\mathrm{GL}_n(F)$  does not depend on the characteristic of the base field  $F$  but only on a certain ratio between  $n$  and  $k$  except in case  $n = 2k$  with odd  $k$ . Moreover, as  $\Gamma(\mathrm{GL}_n(F), X_k) \simeq \Gamma(\mathrm{SL}_n(F), X_k)$  together with the conjugacy class  $X_k$  for even  $1 \leq k \leq n-1$  in case  $\mathrm{char}(F) \neq 2$ , we obtain the same statement for  $G = \mathrm{SL}_n(F)$ . Hence, we provide the exact value of the diameter of the commuting graph for  $\mathrm{SL}_n(F)$ , which improves the main results on the upper bounds of the diameter over  $\mathrm{char}(F) = 2$  in [6, Theorem 1.3].

As an immediate application of Theorem 1.1, we obtain the same result for certain projective special linear groups.

**Corollary 1.2.** *Let  $G = \mathrm{PSL}_n(F)$  over a finite field  $F$  with  $q$  elements and let  $\bar{X}_k$  be the corresponding class of involutions for even  $1 \leq k \leq n-1$  in case  $\mathrm{char}(F) \neq 2$  and  $1 \leq k \leq n/2$  in case  $\mathrm{char}(F) = 2$ . Assume that  $\gcd(n, q-1)$  is odd. Then, for any  $n \geq 3$  and any  $\bar{X}_k$ , the same result holds as in Theorem 1.1*

*Proof.* It suffices to show that  $\Gamma(\mathrm{SL}_n(F), X_k) \simeq \Gamma(\mathrm{PSL}_n(F), \bar{X}_k)$  over a finite field  $F$  with  $q$  elements. Let  $x, y \in X_k$  and  $Z(\mathrm{SL}_n(F)) = \langle a \rangle$ . Assume  $\bar{x} = \bar{y}$  in  $\bar{X}_k$ , i.e.,  $x = a^i y$  for some  $1 \leq i \leq \gcd(n, q-1)$ . Then,  $a^{2i} = 1$ ; thus, by assumption we obtain  $x = y$  in  $X_k$  and  $|X_k| = |\bar{X}_k|$ . Moreover, if  $\bar{x}\bar{y} = \bar{y}\bar{x}$ , i.e.,  $(xy)^2 = a^i I_n$  for some  $1 \leq i \leq \gcd(n, q-1)$ , then  $a^{2i} = 1$ ; thus,  $xy = yx$ . Hence, they have the isomorphic graphs.  $\square$

Applying similar arguments, one can see that the diameter in Theorem 1.1 provides an upper bound of the diameter for certain involution classes of  $\mathrm{PSL}_n(F)$  for even  $n$  and some odd  $q$ .

Now we consider the set  $X$  of all involutions in  $\mathrm{GL}_n(F)$ , i.e.,  $X = \bigcup X_k$ . If  $F$  is finite and  $n \geq 4$ , then a theorem of Brauer-Fowler [3, Theorem (3D)] says  $\mathrm{Diam} \Gamma(\mathrm{GL}_n(F), X) \leq 3$  as mentioned before. For an arbitrary field  $F$ , a similar result [1, Theorem 13] was obtained by Akbari, Mohammadian, Radjavi, and Raja:  $\mathrm{Diam} \Gamma(\mathrm{GL}_n(F), X) = 3$  if  $\mathrm{char}(F) \neq 2$  and  $\mathrm{Diam} \Gamma(\mathrm{GL}_n(F), X) \leq 4$  if  $\mathrm{char}(F) = 2$ . By using the method developed in the proof of the main theorem, we prove

**Corollary 1.3.** *Let  $F$  be a field of characteristic 2,  $n \geq 3$ , and  $X$  the set of all involutions in  $\mathrm{GL}_n(F)$ . Then,  $\mathrm{Diam} \Gamma(\mathrm{GL}_n(F), X) = 3$ .*

Consequently, a theorem of Brauer-Fowler also holds for  $\mathrm{GL}_n(F)$  over an arbitrary field  $F$ . We remark that there have been a number of recent works on the diameter of the commuting graphs of the general

linear groups and matrix algebras over a field. Although we are dealing with a different set of vertices, it would be interesting to compare our result with the results in [1, 7, 11].

As another application of our main theorem, we determine the structure of four dimensional linear group over a finite field (Proposition 3.1 and Proposition 3.5). More precisely, for a fixed involution  $t$  and a class  $X_k$  in  $\text{GL}_4(F)$ , we classify forms of involution  $x \in X_k$  according to the distance  $d(t, x)$  and provide size of  $\Delta_i(t) := \{x \in X_k \mid d(t, x) = i\}$  for each  $i$ . We remark that for a three dimensional linear group, a similar result was previously obtained in [6, Theorem 1.2].

The present paper is organized as follows. In Section 2, we first determine the distance from a fixed involution to an involution in triangular form (Proposition 2.2). Using this result, we then present the proof of our main theorem in Theorem 2.5 and Theorem 2.9 depending on characteristic of the ground field. At the end of this section, we determine the diameter of the graph on the set of all involutions over a field of characteristic 2 (Corollary 2.10). In Section 3, we present the detailed structure of involutions in  $\text{GL}_4(F)$  over a finite field (Proposition 3.1 and Proposition 3.5). In the proof we apply our method developed in the previous section.

In this paper, we denote by  $e_{ij}$  the matrix with a one in position  $i, j$  and zeros elsewhere. For  $a \in F$ , we write  $E_{ij}(a)$  for  $I_n + ae_{ij}$ . We shall use  $M(m, n)$  to denote the set of all  $m \times n$  matrices over a field  $F$ . For any elements  $x, y$  in a group, we simply write  $x \sim y$  if  $xy = yx$ .

## 2. COMMUTING INVOLUTIONS IN GENERAL LINEAR GROUPS

**2.1. The diameter of triangular forms.** For convenience, we shall write  $t$  for  $I(n, k)$ . Then, by a direct computation we see that  $\Delta_1(t)$  consists of partitioned matrices

$$(1) \quad \left( \begin{array}{c|c} A & \\ \hline & C \end{array} \right) \in X_k \text{ (resp. } \left( \begin{array}{c|c} A' & \\ \hline B & C \end{array} \right) \in X_k) \text{ if } \text{char}(F) \neq 2 \text{ (resp. otherwise),}$$

such that  $A' = \left( \begin{array}{c|c} P & Q \\ \hline & C \end{array} \right)$ ,  $A^2 = I_{n-k}$ ,  $C^2 = I_k$ ,  $P^2 = I_{n-2k}$ , and  $A'B = BC$ .

Let  $V$  be an  $n$ -dimensional vector space over  $F$  with the natural action of  $G = \text{GL}_n(F)$ . For an involution  $x \in X_k$  we denote by  $[V, t]$  the image of  $x - 1$ , i.e.,  $[V, x] = \{xv - v \mid v \in V\}$ , thus  $\dim_F[V, x] = k$ . Consider the conjugation action of  $G$  on  $[V, x]$  given by  $g \cdot [V, x] = [V, gxg^{-1}]$ . Note that if  $\{v_1, \dots, v_k\}$  is a basis of  $[V, t]$ , then  $\{gv_1, \dots, gv_k\}$  is a basis of  $g \cdot [V, t]$ . It follows from this that the stabilizer of  $[V, t]$  is given by

$$(2) \quad C_G([V, t]) = \left\{ \left( \begin{array}{c|c} A & \\ \hline B & C \end{array} \right) \mid A \in \text{GL}_{n-k}(F), B \in M(n-k, k), C \in \text{GL}_k(F) \right\};$$

see [6, proof of Theorem 4.6].

Let  $W = [V, x]$  for an involution  $x \in X_k$  and  $m = \dim(W \cap [V, t])$ . Then, we have  $\max\{2k - n, 0\} \leq m \leq k$ . Moreover, it is easy to see that the stabilizer in (2) acts transitively on the collection of such  $k$ -dimensional subspaces  $W$  (see also [6, proof of Theorem 4.6]).

Let  $J(n, k) = \left( \begin{array}{c|c} I_{n-k} & \\ \hline I_k & I_k \end{array} \right)$  for any  $1 \leq k \leq n/2$ . We shall frequently use the following involution in  $X_k$ :

$$(3) \quad t_m = \begin{cases} \left( \begin{array}{c|c} I(n-k, k-m) & \\ \hline & -I(k, k-m) \end{array} \right) & \text{if } \text{char}(F) \neq 2 \\ \left( \begin{array}{c|c|c} I_{n-2k} & & \\ \hline & J(k, k-m) & \\ \hline & & I_{2m-k} & J(k, k-m) \end{array} \right) & \text{if } \text{char}(F) = 2 \text{ and } 2m \geq k \end{cases}$$

such that  $t = w_m t_m w_m$  and  $t \sim t_m$  with  $m = \dim([V, t_m] \cap [V, t])$ , where

$$(4) \quad w_m = \left( \begin{array}{c|c|c|c} I_{n-2k+m} & & & \\ \hline & & & I_{k-m} \\ \hline & & I_m & \\ \hline & I_{k-m} & & \end{array} \right) \left( \text{resp.} \left( \begin{array}{c|c|c|c} I_{n-2k+m} & & & \\ \hline & & I_{k-m} & \\ \hline & I_{k-m} & & \\ \hline & & & I_m \end{array} \right) \right)$$

in the corresponding cases. Similarly, we define  $t_m$  to be the transpose of the second involution  $t_{k-m}$  in (3) if  $\text{char}(F) = 2$  and  $2m < k$ . In this case, we have  $t^T = w_{k-m} t_m w_{k-m}$  and  $t^T \sim t_m$ .

**Lemma 2.1.** (cf. [6, Lemma 4.1]) *Let  $t \neq x \in X_k$  with  $[V, t] = [V, x]$ . Then,  $x$  is of the form  $x = \left( \begin{array}{c|c} I_{n-k} & \\ \hline A & -I_k \end{array} \right)$  for some  $A \in M(k, n-k)$  if  $\text{char}(F) \neq 2$  and  $x = \left( \begin{array}{c|c} I_{n-k} & \\ \hline B & I_k \end{array} \right)$  for some  $B \in M(k, n-k)$  with  $\text{rank}(B) = k$  otherwise. In particular,  $d(t, x) = 1$  in the latter case.*

*Proof.* Let  $x = gtg^{-1}$  for some  $g \in \text{GL}_n(F)$ . Then, by assumption we obtain  $g \cdot [V, t] = [V, t]$ , thus  $g \in C_G([V, t])$ . As  $g$  is a block lower triangular as in (2), the result immediately follows. The second statement follows from (1).  $\square$

We first determine the distance from  $t$  to an involution which is a block lower triangular.

**Proposition 2.2.** *Let  $x = \left( \begin{array}{c|c} A & \\ \hline B & C \end{array} \right) \in X_k$  with  $A \in \text{GL}_{n-k}(F)$ ,  $B \in M(k, n-k)$ ,  $C \in \text{GL}_k(F)$ . Then,  $d(t, x) \leq 3$  if  $\text{char}(F) \neq 2$  and  $n = 2k$  with  $k$  odd and  $d(t, x) \leq 2$  otherwise.*

*Proof.* First consider the case that  $\text{char}(F) = 2$ . Let  $y = \left( \begin{array}{c|c} \frac{J(n-k, k_1)}{D} & \\ \hline & J(k, k_2) \end{array} \right) \in X_k$  for some  $k_1, k_2 \geq 0$  and  $D \in M(k, n-k)$ . We first show that  $k_1 \geq k_2$ . Since  $DJ(n-k, k_1) = J(k, k_2)D$ , we see that  $D = \left( \begin{array}{c|c} D_1 & \\ \hline D_3 & D_4 \end{array} \right)$  for some  $D_1 \in M(k-k_2, k_1)$ ,  $D_2 \in M(k-2k_2, n-k-2k_1)$ ,  $D_3 \in M(k_2, k_1)$ ,  $D_4 \in M(k_2, n-k-k_1)$ . As  $y \in X_k$ , we have  $k_1 \geq k_2$ , thus we can find a block diagonal matrix  $g \in C_G([V, t])$  such that

$$y = g \left( \begin{array}{c|c|c} \frac{J(n-2k, k_1-k_2)}{D'} & & \\ \hline & J(k, k_2) & \\ \hline & & J(k, k_2) \end{array} \right) g^{-1} =: gzg^{-1}$$

for some  $D' \in M(k, n-k)$ . Let  $x = \left( \begin{array}{c|c} A & \\ \hline B & C \end{array} \right) \in X_k$  for some  $A \in \text{GL}_{n-k}(F)$ ,  $C \in \text{GL}_k(F)$ . As  $A^2 = I_{n-k}$  and  $C^2 = I_k$ , there exists a block diagonal matrix  $h \in C_G([V, t])$  such that  $x = hyh^{-1} = hgz(hg)^{-1}$ . Since  $z \in \Delta_1(t)$ , it follows from Lemma 2.1 that  $d(t, x) \leq d(t, hgt(hg)^{-1}) + d(hgt(hg)^{-1}, hgz(hg)^{-1}) \leq 1 + 1 = 2$ .

Consider the case that  $\text{char}(F) \neq 2$ . Let  $y' = \left( \begin{array}{c|c} \frac{-I(n-k, n+m-2k)}{E} & \\ \hline & I(k, m) \end{array} \right) \in X_k$  for some  $0 \leq m \leq k$  and  $E \neq 0$ . Since  $(y')^2 = I_n$ , we see that the matrix  $E$  is of the form  $E = \left( \begin{array}{c|c} E_1 & \\ \hline & E_2 \end{array} \right)$  for some  $E_1 \in M(k-m, k-m)$  and  $E_2 \in M(m, n+m-2k)$ . Let  $P_i, Q_i$ ,  $1 \leq i \leq 2$ , be products of elementary matrices such that

$$(5) \quad P_i E_i Q_i = \left( \begin{array}{c|c} I_{l_i} & \\ \hline & \end{array} \right), \text{ where } l_i = \text{rank}(E_i)$$

and let  $g' = \left( \begin{array}{c|c|c|c} Q_1 & & & \\ \hline & Q_2 & & \\ \hline & & P_1^{-1} & \\ \hline & & & P_2^{-1} \end{array} \right)$ . Consider a diagonal matrix  $t' (\neq t)$  obtained by permuting the diagonal entries of  $t$ , written as  $t' = \left( \begin{array}{c|c|c|c} R_1 & & & \\ \hline & R_2 & & \\ \hline & & S_1 & \\ \hline & & & S_2 \end{array} \right)$  with  $R_1, S_1 \in M(k-m, k-m)$ ,  $S_2 \in M(m, m)$ ,  $R_2 \in M(n+m-2k, n+m-2k)$  such that

$$(6) \quad S_i(P_i E_i Q_i) = (P_i E_i Q_i) R_i.$$

This is guaranteed by (5). Note that  $n = 2k$  with  $k = l_1 + l_2$  odd if and only if there is no  $t' \in X_k$ . To see the forward direction, assume  $t' \in X_k$ . Then, the assumption and (5) implies  $R_i = S_i$  for all  $i$ , which is impossible. Similarly, one can easily check the reverse direction.

If  $t' \in X_k$ , then by (6) we see that  $y'$  commutes with  $g't'(g')^{-1} \in \Delta_1(t)$ . As  $x = h'y'(h')^{-1}$  for some block diagonal matrix  $h' \in C_G(t)$ , we conclude that  $x$  commutes with  $(h'g')t'(h'g')^{-1} \in \Delta_1(t)$ , i.e.,  $d(t, x) = 2$ .

Otherwise, we see that our matrix

$$y' = \left( \begin{array}{c|c|c|c} -I_{k_1} & & & \\ \hline & I_{k_2} & & \\ \hline E_1 & & I_{k_1} & \\ \hline & E_2 & & -I_{k_2} \end{array} \right),$$

where  $k_1 = k - m, k_2 = m$ , is a composition of 2 submatrices

$$y'_1 = \left( \begin{array}{c|c} -I_{k_1} & \\ \hline E_1 & I_{k_1} \end{array} \right), y'_2 = \left( \begin{array}{c|c} I_{k_2} & \\ \hline E_2 & -I_{k_2} \end{array} \right).$$

Therefore, it is enough to check that  $d(t_i, y'_i) \leq 3$  for each  $1 \leq i \leq 2$ , where  $t_i = I(2k_i, k_i)$ . We shall consider only the case  $i = 1$  because the other case is similar. Thus, we may assume that  $k = k_1$  is odd,  $y' = y'_1$ , and  $t = t_1$ . Write  $y' = u \left( \begin{array}{c|c} -I_k & \\ \hline I_k & I_k \end{array} \right) u^{-1} =: uz'u^{-1}$  for some block diagonal matrix  $u \in C_G(t)$ . As  $z'$  commutes with  $x' := \left( \begin{array}{c|c} I(k, [k/2]) & \\ \hline U & I(k, [k/2]) \end{array} \right)$ , where  $U = (I(k, [k/2]) - I(k, [k/2]))/2$ ,  $x$  commutes with  $(h'u)x'(h'u)^{-1}$ . Since  $\text{rank}(U) \neq k$ , it follows from the previous result that  $d(t, (h'u)x'(h'u)^{-1}) = 2$ , thus  $d(t, x) \leq 3$ . Indeed,  $d(t, x) = 3$  as  $z'$  does not commute with any element in  $\Delta_1(t)$ .  $\square$

**Remark 2.3.** By the same argument as in the proof of Proposition 2.2, we see that for any  $x \in X_k$  of the form  $\left( \begin{array}{c|c} A & B \\ \hline C & \end{array} \right)$ ,  $A \in \text{GL}_{n-k}(F)$ ,  $C \in \text{GL}_k(F)$ ,  $B \neq 0$ ,  $d(t, x) \leq 3$  in case  $\text{char}(F) \neq 2$ . More precisely,  $d(t, x) \leq 3$  if  $n = 2k$  with  $k$  odd and  $d(t, x) \leq 2$  otherwise.

**2.2. The diameter in characteristic different from 2.** In [1, Theorem 10], it was shown that for any  $k = [n/2]$   $\text{Diam } \Gamma(G, X_k) \geq 3$ . We first present a general lower bound for the diameter in case  $\text{char}(F) \neq 2$ . As  $\Gamma(G, X_k) \simeq \Gamma(G, X_{n-k})$  by the map  $x \mapsto -x$ , we shall only consider the case where  $n \geq 2k$ .

**Proposition 2.4.** *Let  $F$  be a field of characteristic different from 2 and  $n \geq 3$ . Then, for any integer  $k$  with  $n < 4k < 3n$ ,  $\text{Diam } \Gamma(G, X_k) \geq 3$ .*

*Proof.* It suffices to consider the case  $2k < n < 4k$ . We shall find  $x \in X_k$  such that it does not commute with any involution in  $\Delta_1(t)$ . Let  $A = \left( \begin{array}{c|c} & \\ \hline I_{k-1} & \end{array} \right) - 2I_k \in \text{GL}_k(F)$ . Then, obviously, the only involution commuting with  $A$  is  $-I_k$ . We divide the proof into the following two cases.

Assume  $2k < n < 3k$ . Consider the following involution in  $X_k$

$$x = g \left( \begin{array}{c|c|c} I_{n-2k} & & \\ \hline & -I_k & I_k \\ \hline & & I_k \end{array} \right) g^{-1} \text{ with } g = \left( \begin{array}{c|c|c} I_{n-2k} & & \\ \hline & I_k & \\ \hline & A & I_k \end{array} \right).$$

If  $d(t, x) = 2$ , then by (1) there exists

$$y = g \left( \begin{array}{c|c|c} B & & \\ \hline C & D & \\ \hline 2C & & D \end{array} \right) g^{-1} \in \Delta_1(t)$$

such that  $B^2 = I_{n-2k}$ ,  $D^2 = I_k$ , and  $AD = DA$ . Hence, by assumption we have  $D = I_k$ , which contradicts  $y \in X_k$ . Therefore,  $d(t, x) \geq 3$ .

Now we assume that  $3k \leq n < 4k$ . Consider an involution in  $X_k$

$$x = g \left( \begin{array}{c|c|c|c} I_{n-2k} & & & \\ \hline & I_k & -I_k & I_k \\ \hline & & & I_k \end{array} \right) g^{-1} \text{ with } g = \left( \begin{array}{c|c|c|c} I_{n-2k} & & & \\ \hline & -I_k & I_k & \\ \hline & I_k & A & I_k \end{array} \right)$$

Similarly, if  $d(t, x) = 2$ , then by (1) there exists

$$y = g \left( \begin{array}{c|c|c} B & & \\ \hline C & I_k & \\ \hline D & & I_k \end{array} \right) g^{-1} \in \Delta_1(t) \text{ with } B^2 = I_{n-2k}$$

such that  $CB + C = DB + D = 0$ ,

$$(7) \quad C = \frac{1}{2} \left( \begin{array}{c|c} B_3 + D_1 & B_4 + D_2 - I_k \end{array} \right), \text{ and } \left( \begin{array}{c} I_k \end{array} \right) (B - I_{n-2k}) + A(C + \left( \begin{array}{c} I_k \end{array} \right)) + D = 0,$$

where  $B = \left( \begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array} \right)$  and  $D = \left( \begin{array}{c} D_1 \\ D_2 \end{array} \right)$  with  $B_4, D_2 \in M(k, k)$ .

Write  $B = PI(n-2k, k)P^{-1}$  for some  $P = \left( \begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array} \right) \in GL_{n-2k}(F)$  with  $P_4 \in M(k, k)$ . Then, it follows by (7) that  $P_3 = 0$  and

$$\left( \begin{array}{c} -2I_k \end{array} \right) = \left( \begin{array}{c} \frac{1}{2}A(B_3 + D_1) - D_1 \\ -\frac{1}{2}AD_2 - D_2 \end{array} \right).$$

Therefore, we obtain  $4I_k = (A + 2I_k)D_2$ , which is impossible as  $A + 2I_k$  is singular. Hence,  $d(t, x) \geq 3$ .  $\square$

Applying Propositions 2.2 and 2.4, we first determine the diameter in case  $\text{char}(F) \neq 2$ .

**Theorem 2.5.** *Let  $F$  be a field of characteristic different from 2 and  $n \geq 3$ . Then, for any integer  $k$  with  $n < 4k < 3n$ ,  $\text{Diam } \Gamma(G, X_k) = 3$ .*

*Proof.* It is enough to prove that the lower bounds in Proposition 2.4 are tight. We first assume that  $n > 2k$  or  $n = 2k$  with  $k$  even. Let  $x \in X_k$  with  $\dim([V, t] \cap [V, x]) = m$ . By transitivity, there is a block lower triangular matrix  $g$  such that  $[V, x] = [V, gt_m g^{-1}] = [V, gw_m tw_m g^{-1}]$ . Therefore, we have

$$w_m g^{-1} x g w_m = \left( \begin{array}{c|c} I_{n-k} & \\ \hline B & -I_k \end{array} \right)$$

for some  $B \in M(k, n-k)$ . Let  $B = \left( \begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array} \right)$  with  $B_1 \in M(m, n-2k+m)$ ,  $B_4 \in M(k-m, k-m)$ .

Then, the product  $g^{-1}xg$  is decomposed as

$$h' \left( \begin{array}{c|c|c|c} I_{n-2k+m} & & & \\ \hline & -I_{k-m} & & B_4 \\ \hline & & -I_m & \\ \hline & & & I_{k-m} \end{array} \right) h'^{-1}, \text{ where } h' = \left( \begin{array}{c|c|c|c} I_{n-2k+m} & & & \\ \hline B_3/2 & I_{k-m} & & \\ \hline B_1/2 & & I_m & B_2/2 \\ \hline & & & I_{k-m} \end{array} \right).$$

Let  $y$  denote the middle matrix in the above decomposition and  $h = gh'$ . Then,  $x = hyh^{-1}$ . By Remark 2.3, there exists  $z \in \Delta_1(t)$  such that  $yz = zy$ . Hence,  $hzh^{-1}$  commutes with  $x$ . As  $hzh^{-1}$  is block lower triangular, it follows from Proposition 2.2 that  $d(hzh^{-1}, t) \leq 2$ . Therefore,  $d(t, x) \leq 3$ .

Now we consider the case where  $n = 2k$  and  $k$  odd. Note that a conjugation on  $x$  by a block diagonal matrix in  $C_G(t)$  does not affect on the distance from  $t$ , i.e., all  $C_G(t)$ -conjugacy class of  $x$  has the same distance from  $t$ . Hence, after a suitable modification, we may assume that

$$(8) \quad x = \left( \begin{array}{c|c} I_k & \\ \hline L & I_k \end{array} \right) \left( \begin{array}{c|c|c|c} I_m & & & \\ \hline & -I_{k-m} & & D \\ \hline & & -I_m & \\ \hline & & & I_{k-m} \end{array} \right) \left( \begin{array}{c|c} I_k & \\ \hline -L & I_k \end{array} \right), \text{ where } D = \left( \begin{array}{c|c} I_l & \\ \hline & 0 \end{array} \right)$$

for some  $0 \leq l \leq k - m$  and  $L \in M(k, k)$ . We shall continue to denote by  $h$  and  $y$  the first and the second matrices in (8), respectively. If  $l = 0$ , then  $x$  is block lower triangular, thus by Proposition 2.2, we have  $d(t, x) \leq 3$ .

Assume that  $m = 0$  and both  $L$  and  $D$  are singular ( $0 < l < k$ ). Observe that for any  $M = \left( \begin{array}{c|c} M_1 & \\ \hline & M_2 \end{array} \right)$  and  $N = \left( \begin{array}{c|c} M_1 & \\ \hline N_2 & N_3 \end{array} \right)$  with  $M_1 \in \text{GL}_l(F)$ ,  $M_2, N_3 \in \text{GL}_{k-l}(F)$ , the matrix  $x = hyh^{-1}$  in (8) is  $C_G(t)$ -conjugate to  $\left( \begin{array}{c|c} I_k & \\ \hline L' & I_k \end{array} \right) y \left( \begin{array}{c|c} I_k & \\ \hline -L' & I_k \end{array} \right)$ , where  $L' = N^{-1}LM$ , thus we may replace  $L$  by  $L'$ . Write  $L = \left( \begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array} \right)$ ,  $L_1 \in M(l, l)$ ,  $L_4 \in M(k-l, k-l)$  and  $L' = (L'_i)$  in the same form. If  $L_2 \neq 0$ , then we choose  $M_1$  and  $M_2$  such that the last column of  $L'_2 = M_1^{-1}L_2M_2$  is  $(0, \dots, 0, 1)^T$ . Let  $v$  be the last column of  $L_4M_2$ . Set  $N_2 = \left( \begin{array}{c|c} & \\ \hline & v \end{array} \right)$ . Then, all entries of the last column of  $L'_4 = -N_2M_1^{-1}M_2 + L_4M_2$  are zero except the  $l$ -th element. Choose a diagonal matrix  $t'$  obtained by permuting the diagonal entries of  $t$  whose  $l$ -th,  $k$ -th, and  $(l+k)$ -th entries are all  $-1$ . Then, we see that  $y \sim t'$  and the  $(2, 1)$ -block of  $ht'h^{-1}$  is singular. Hence by Proposition 2.2, we obtain  $d(t, x) = d(t, ht'h^{-1}) + d(ht'h^{-1}, x) \leq 3$ . Similarly, if  $L_2 = 0$ , then we can find  $M$  and  $N$  as above such that  $L' = \left( \begin{array}{c|c} J_1 & \\ \hline L'_3 & L'_4 \end{array} \right)$ , where  $J_1$  is a rational canonical form of  $L_1$  and  $L'_4 = \left( \begin{array}{c|c} & \\ \hline & I_{l'} \end{array} \right)$  for some  $0 \leq l' \leq k-l$ . If  $J_1$  is singular, then the  $(2, 1)$ -block of  $ht'h^{-1}$  is singular, thus we may assume that  $J_1$  is nonsingular. Moreover, by applying appropriate  $M$  and  $N$ , we may assume that  $L'_3 = 0$ , thus  $L'$  has a zero row, which implies that the  $(2, 1)$ -block of  $ht'h^{-1}$  is singular and  $d(t, x) \leq 3$ . We divide the remaining proof into the following three cases.

Case:  $m = 0$  and  $L \in \text{GL}_k(F)$  or  $m = 0$ ,  $D = I_k$ , and  $L \notin \text{GL}_k(F)$ . We start with the first case. Write  $L = QP^{-1}$  for some  $P, Q \in \text{GL}_k(F)$ . Then,  $x$  is  $C_G(t)$ -conjugate to

$$(9) \quad \left( \begin{array}{c|c} I_k & \\ \hline I_k & I_k \end{array} \right) \left( \begin{array}{c|c} -I_k & J \\ \hline & I_k \end{array} \right) \left( \begin{array}{c|c} I_k & \\ \hline -I_k & I_k \end{array} \right), \text{ where } J = \left( \begin{array}{c|c} J_1 & \\ \hline & J_2 \end{array} \right)$$

is a rational canonical form of  $P^{-1}DQ$  with a singular part  $J_1 \in M(k_1, k_1)$  and a nonsingular part  $J_2 \in \text{GL}_{k_2}(F)$ .

Let  $A, B \in \text{GL}_{k_2}(F)$  such that  $AJ_2B^{-1} = I_{k_2}$ . Then, the matrix in (9) is  $C_G(t)$ -conjugate to a matrix

$$(10) \quad \left( \begin{array}{c|c|c|c} I_{k_1} & & & \\ \hline & I_{k_2} & & \\ \hline I_{k_1} & & I_{k_1} & \\ \hline & J' & & I_{k_2} \end{array} \right) \left( \begin{array}{c|c|c|c} -I_{k_1} & & J_1 & \\ \hline & -I_{k_2} & & I_{k_2} \\ \hline & & I_{k_1} & \\ \hline & & & I_{k_2} \end{array} \right) \left( \begin{array}{c|c|c|c} I_{k_1} & & & \\ \hline & I_{k_2} & & \\ \hline -I_{k_1} & & I_{k_1} & \\ \hline & -J' & & I_{k_2} \end{array} \right)$$



where  $J'$  is a rational canonical form of  $BA^{-1}$ . As in the proof of Proposition 2.2, the matrix in (10) is a composition of following two submatrices

$$(11) \quad x_1 = \left( \begin{array}{c|c} I_{k_1} & \\ \hline I_{k_1} & I_{k_1} \end{array} \right) \left( \begin{array}{c|c} -I_{k_1} & J_1 \\ \hline & I_{k_1} \end{array} \right) \left( \begin{array}{c|c} I_{k_1} & \\ \hline -I_{k_1} & I_{k_1} \end{array} \right), \quad x_2 = \left( \begin{array}{c|c} I_{k_2} & \\ \hline J' & I_{k_2} \end{array} \right) \left( \begin{array}{c|c} -I_{k_2} & I_{k_2} \\ \hline & I_{k_2} \end{array} \right) \left( \begin{array}{c|c} I_{k_2} & \\ \hline -J' & I_{k_2} \end{array} \right),$$

it is enough to show that  $d(t_i, x_i) \leq 3$  for all  $1 \leq i \leq 2$ , where  $t_i = I(2k_i, k_i)$ . For the same reason, we may assume that  $J_1$  is a singular companion matrix and  $J'$  is one of the followings: a diagonal matrix, a nonsingular companion matrix or a direct sum of an element in  $F^\times$ , and a nonsingular companion matrix. If  $k_i$  is even for some  $i$ , then it follows by the previous case that  $d(t_i, x_i) \leq 3$ . Therefore, it suffices to consider the distance for each odd  $k_i$ .

Let  $k_i = 2r_i + 1$  for  $1 \leq i \leq 2$ . Consider the following involution in  $X_{k_1}$

$$y_1 = \left( \begin{array}{c|c} I_{k_1} & \\ \hline I_{k_1} & I_{k_1} \end{array} \right) \left( \begin{array}{c|c} Y & Z \\ \hline & W \end{array} \right) \left( \begin{array}{c|c} I_{k_1} & \\ \hline -I_{k_1} & I_{k_1} \end{array} \right), \quad \text{where } Z = \frac{J_1 W - Y J_1}{2},$$

$$Y = \left( \begin{array}{c|c|c} I(r_1, r_1 - 1) & & \\ \hline & -1 & \\ \hline & u & I_{r_1} \end{array} \right), \quad W = \left( \begin{array}{c|c|c} I_{r_1} & & \\ \hline v & -1 & \\ \hline & & -I_{r_1} \end{array} \right), \quad u = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v = (0 \quad \cdots \quad 0 \quad -1).$$

Using the fact that the first row of  $J_1$  is zero, one can easily check that

$$x_1 \sim y_1 \sim \left( \begin{array}{c|c} I(k_1, r_1) & \\ \hline & I(k_1, r_1 + 1) \end{array} \right) \in \Delta_1(t_1),$$

thus  $d(t_1, x_1) \leq 3$  for odd  $k_1$ .

Now we show that  $d(t_2, x_2) \leq 3$  for odd  $k_2$ . Assume that  $J'$  is a diagonal matrix. Then, we have

$$(12) \quad \left( \begin{array}{c|c} -I_{k_2} & I_{k_2} \\ \hline & I_{k_2} \end{array} \right) \sim \left( \begin{array}{c|c} I(k_2, r_2) & -e_{(r_2+1)(r_2+1)} \\ \hline & I(k_2, r_2 + 1) \end{array} \right) \sim \left( \begin{array}{c|c} Y & \\ \hline W J' - J' Y & W \end{array} \right),$$

where

$$Y = I(k_2, r_2), \quad W = \left( \begin{array}{c|c} -1 & \\ \hline & I(2r_2, r_2) \end{array} \right).$$

Hence, after the conjugation by the first matrix in the decomposition of  $x_2$  we obtain  $d(t_2, x_2) \leq 3$ . Similarly, if  $J'$  is either a nonsingular companion matrix or a direct sum of an element in  $F^\times$  and a nonsingular companion matrix, then the same commutativity (12) holds if we replace  $Y$  and  $W$  by

$$(13) \quad Y = \left( \begin{array}{c|c|c} I(r_2, r_2 - 1) & & \\ \hline & -1 & v \\ \hline & & I_{r_2} \end{array} \right), \quad W = \left( \begin{array}{c|c|c} I_{r_2} & & \\ \hline & -1 & \\ \hline & & -I_{r_2} \end{array} \right)$$

where  $v = (v_1 \quad \cdots \quad v_{r_2})$  with  $v_i = -(J')_{r_2+1, r_2+1+i}$  for all  $1 \leq i \leq r_2$ . Hence,  $d(t_2, x_2) \leq 3$ . For the second case, we may replace the matrix  $L$  in (8) by its rational canonical form. Then, the same proof above works with  $Y$  and  $W$  in (12) and (13), replacing  $k_2$  by  $k$ .

**Case:  $k > 3$  and  $m > 0$ .** Write  $L = \left( \begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array} \right)$  with  $L_1 \in M(m, m), L_4 \in M(k-m, k-m)$ . Find  $M, N \in GL_m(F)$  and a lower triangular matrix  $P \in GL_{k-m}(F)$  such that  $M^{-1}L_1N$  is a diagonal matrix and the last column of  $P^{-1}L_3N$  has at most one nonzero element in the  $i_0$ -th row. Then, we see that  $x$  is  $C_G(t)$ -conjugate to

$$(14) \quad \left( \begin{array}{c|c} I_k & \\ \hline L' & I_k \end{array} \right) \left( \begin{array}{c|c|c|c} I_m & & & \\ \hline & -I_{k-m} & & D' \\ \hline & & -I_m & \\ \hline & & & I_{k-m} \end{array} \right) \left( \begin{array}{c|c} I_k & \\ \hline -L' & I_k \end{array} \right),$$



where  $P' \in \text{GL}_l(F)$  is the (1,1)-block of  $P$ ,

$$L' = \left( \begin{array}{c|c} M & \\ \hline & P \end{array} \right)^{-1} L \left( \begin{array}{c|c} N & \\ \hline & I_{k-m} \end{array} \right) \text{ and } D' = DP = \left( \begin{array}{c|c} P' & \\ \hline & \end{array} \right).$$

Hence, we may replace  $x$  by the matrix in (14). We shall denote by  $h'$  and  $y'$  the first and the second matrices in (14), respectively.

We choose diagonal matrices  $Y_i \in \text{GL}_m(F)$ ,  $W_i \in \text{GL}_{k-m}(F)$  for  $1 \leq i \leq 2$  such that the last diagonal entries of  $Y_i$  and the  $i_0$ -th diagonal entry of  $W_2$  are all  $-1$ ,  $DW_2 = W_1D$ , and

$$z := \left( \begin{array}{c|c|c|c} Y_1 & & & \\ \hline & W'_1 & & \\ \hline & & Y_2 & \\ \hline & & & W_2 \end{array} \right) \in X_k \text{ with } W'_1 = \left( \begin{array}{c|c} P' & \\ \hline & I_{k-m-l} \end{array} \right) W_1 \left( \begin{array}{c|c} P' & \\ \hline & I_{k-m-l} \end{array} \right)^{-1}$$

Then, we have  $z \sim y'$  and the  $m$ -th column of (2,1)-block of  $h'zh'^{-1}$  is zero, thus by Proposition 2.2  $d(t, h'zh'^{-1}) \leq 2$ , so  $d(t, x) \leq 3$ .

Case:  $k = 3$  and  $m > 0$  Assume that  $m = 2$ . Then, by the same argument as in the previous case, we may assume that

$$x = \left( \begin{array}{c|c} I_3 & \\ \hline & I_3 \end{array} \right) \left( \begin{array}{c|c|c|c} I_2 & & & \\ \hline & -1 & & 1 \\ \hline & & -I_2 & \\ \hline & & & 1 \end{array} \right) \left( \begin{array}{c|c} I_k & \\ \hline -L & I_k \end{array} \right) =: hyh^{-1}, \text{ with } L = \left( \begin{array}{c|c|c} * & * & * \\ \hline & * & * \\ \hline * & & * \end{array} \right) \text{ or } \left( \begin{array}{c|c|c} * & & * \\ \hline * & * & * \\ \hline & * & * \end{array} \right).$$

Then,  $y$  commutes with the diagonal matrix  $z$  with entries  $(-1, 1, -1, 1, 1, -1)$  and  $(1, -1, -1, 1, 1, -1)$ . As the last row of (2,1)-block of  $hzh^{-1}$  is zero, it follows from Proposition 2.2 that  $d(t, x) \leq 3$ .

Now we assume that  $m = 1$ . First of all, if either the first row or the second row in the (2,2)-block of  $L$  (i.e.,  $L_4 \in \text{M}(2, 2)$ ) is zero, then the middle matrix  $y$  in (8) commutes with the diagonal matrices  $z_1 = \text{diag}(-1, -1, 1, 1, -1, 1)$  or  $z_2 = \text{diag}(-1, 1, -1, 1, 1, -1)$ . Moreover, the (2, 1)-block of  $hz_ih^{-1}$  is singular for each  $1 \leq i \leq 2$ . Therefore, by Proposition 2.2 we conclude that  $d(t, x) \leq 3$ . Hence, after a suitable modification, we may assume that  $L_4$  is invertible.

If  $D$  is invertible (i.e.,  $D = I_2$ ) then, we have

$$x = \left( \begin{array}{c|c} I_3 & \\ \hline & I_3 \end{array} \right) \left( \begin{array}{c|c|c|c} 1 & & & \\ \hline & -I_2 & & I_2 \\ \hline & & -1 & \\ \hline & & & I_2 \end{array} \right) \left( \begin{array}{c|c} I_3 & \\ \hline -L & I_3 \end{array} \right) =: hyh^{-1}, \text{ where } L = \left( \begin{array}{c|c|c} i & a & b \\ \hline c & & d \\ \hline e & 1 & f \end{array} \right)$$

with  $i = 0$  or  $1$  and  $d \neq 0, a, b, c, e, f \in F$  (if  $L_4$  is diagonalizable, then by the same argument as in the case  $m = 2$  we obtain  $d(t, x) \leq 3$ ). One can easily verify that

$$y \sim \left( \begin{array}{c|c} Y & \\ \hline Z & W \end{array} \right) =: z,$$

where

$$(15) \quad Y = \left( \begin{array}{c|c|c} 1 & & \\ \hline & 1 & \\ \hline q & p & -1 \end{array} \right), W = \left( \begin{array}{c|c|c} -1 & r & \\ \hline & 1 & \\ \hline & p & -1 \end{array} \right), Z = \left( \begin{array}{c|c} & -2r \\ \hline 2q & \end{array} \right)$$

$$\text{with } (p, q, r) = \begin{cases} (2e/c, 0, 0) & \text{if } i = 0, c \neq 0, \\ (-2a/b, 0, 0) & \text{if } i = 0, c = 0, b \neq 0, \\ (2e/c, 0, 2/c) & \text{if } i = 1, c \neq 0, \\ (-2a/b, -2/b, 0) & \text{if } i = 1, c = 0, b \neq 0, \\ ((2a - 4)/d, 2/d, 2) & \text{if } i = 1, c = b = 0. \end{cases}$$

If  $i = c = b = 0$ , then take  $z = \text{diag}(-1, 1, -1, 1, 1, -1)$ . As the  $(2, 1)$ -block of  $hzh^{-1}$  is singular, it follows from Proposition 2.2 that  $d(t, x) \leq 3$ .

Finally, if  $D$  is singular (i.e.,  $D = e_{11}$ ) and  $L_4$  is nonsingular, then  $x$  is  $C_G(t)$ -conjugate to

$$(16) \quad x' = \left( \begin{array}{c|c} I_3 & \\ \hline L' & I_3 \end{array} \right) \left( \begin{array}{c|c|c|c} 1 & & & \\ \hline & -I_2 & & D' \\ \hline & & -1 & \\ \hline & & & I_2 \end{array} \right) \left( \begin{array}{c|c} I_3 & \\ \hline -L' & I_3 \end{array} \right), \text{ where } L' = \left( \begin{array}{c|c|c} i & a & b \\ \hline c & 1 & \\ \hline d & & 1 \end{array} \right), D' = \left( \begin{array}{cc} & \\ 1 & e \end{array} \right)$$

with  $0 \leq i \leq 1$  and  $a, b, c, d, e \in F$ . We denote by  $h'$  and  $y'$  the first and the second matrices in (16), respectively. Consider an involution  $z'$  in  $X_3$

$$(17) \quad z' = \begin{cases} \text{diag}(1, 1, -1, 1, -1, -1) & \text{if } d = 0, \\ \text{diag}(1, -1, -1, 1, -1, 1) & \text{if } c = e = 0, \\ \text{diag}(-1, -1, 1, 1, -1, 1) + (2/e)e_{65} & \text{if } c = 0, e \neq 0. \end{cases}$$

Then, we see that  $y' \sim z'$  and the  $(2, 1)$ -block of  $h'z'h'^{-1}$  is singular, thus  $d(t, x) \leq 3$  in corresponding cases. If  $c, d \in F^\times$ , then we replace  $x'$  in (16) by the following  $C_G(t)$ -conjugate involution  $x''$

$$\left( \begin{array}{c|c} I_3 & \\ \hline L'' & I_3 \end{array} \right) \left( \begin{array}{c|c|c|c} 1 & & & \\ \hline & -I_2 & & D'' \\ \hline & & -1 & \\ \hline & & & I_2 \end{array} \right) \left( \begin{array}{c|c} I_3 & \\ \hline -L'' & I_3 \end{array} \right), \text{ where } L'' = \left( \begin{array}{c|c|c} i & a & b \\ \hline c & 1 & \\ \hline -d/c & & 1 \end{array} \right), D'' = \left( \begin{array}{c|c} & \\ \hline 1 + de/c & e \end{array} \right).$$

Applying the same argument together with  $z' = \text{diag}(1, 1, -1, 1, -1, -1)$ , we conclude that  $d(t, x) \leq 3$ .  $\square$

In the following corollary, we further reduce the upper bound for  $\text{Diam } \Gamma(G, X_k)$  over a field  $F$  of characteristic different 2. The same result in case  $\text{char}(F) = 2$  was obtained in [6, Lemma 4.3].

**Corollary 2.6.** *Let  $F$  be a field of characteristic different 2 and  $n \geq 3$ . Then,  $\text{Diam } \Gamma(G, X_k) = 2$  if  $n \geq 4k$  or  $n \geq 4(n - k)$ .*

*Proof.* Let  $x = hyh^{-1}$  as in the proof of Theorem 2.5. Write  $h = u \left( \begin{array}{c|c|c} I_{n-k} & & \\ \hline Q_1 & Q_2 & I_k \end{array} \right) =: uv$  for some block diagonal matrix  $u \in C_G(t)$ ,  $Q_1 \in M(k, n - 2k + m)$ , and  $Q_2 \in M(k, k - m)$ . Hence,  $x = uvv(uv)^{-1}$ .

We find  $z \in \Delta_1(t)$  such that  $yz = zy$  and  $vzv^{-1} \in \Delta_1(t)$ . By assumption that  $n - 3k + m \geq k$ , we can find a matrix  $P \in \text{GL}_{n-2k+m}(F)$  such that the first  $k$  columns of  $Q_1P$  are all zero. Take

$$z = \left( \begin{array}{c|c|c} C & & \\ \hline & I_{k-m} & \\ \hline & & I_k \end{array} \right)$$

where  $C = -PI(n - 2k + m, n - 3k + m)P^{-1}$ . Obviously,  $z$  commutes with  $y$ . Moreover, by a direct computation we obtain  $Q_1C = Q_1$ , thus  $vzv^{-1} \in \Delta_1(t)$ . Therefore,  $uvz(uv)^{-1} \in \Delta_1(t)$  and  $d(t, x) = d(t, uvz(uv)^{-1}) + d(uvz(uv)^{-1}, x) = 2$ .  $\square$

**2.3. The diameter in characteristic 2.** We first provide an involution which has diameter at least 4.

**Lemma 2.7.** *Let  $F$  be a field of characteristic 2 and  $n \geq 3$ . Then,  $\text{Diam } \Gamma(G, X_k) \geq 4$  if  $n = 2k$  with  $k$  odd.*

*Proof.* We show that  $d(t, t^T) \geq 4$ . By (1), it is enough to show that  $d(t, t^T) \neq 3$ . Assume that  $d(t, t^T) = 3$ , i.e., there exist commuting involutions  $x \in \Delta_1(t)$  and  $y \in \Delta_1(t^T)$ . As  $x, y \in X_k$  and  $xy = yx$ , we may assume that

$$x = \left( \begin{array}{c|c} J(k, r) & \\ \hline B & J(k, r) \end{array} \right) \text{ and } y = \left( \begin{array}{c|c} P & Q \\ \hline & P \end{array} \right)$$

for some involution  $P \in \text{GL}_k(F)$  and  $1 \leq r \leq [k/2]$ . Note that the centralizer  $C_{M(k,k)}(J(k,r))$  consists of

$$\left( \begin{array}{c|c|c} B_1 & & \\ \hline B_2 & B_3 & \\ \hline B_4 & B_5 & B_1 \end{array} \right) \text{ with } B_1 \in M(r,r), B_3 \in \text{GL}_{k-2r}(F).$$

Thus, the matrix  $Q = (Q_i)$ ,  $1 \leq i \leq 5$ , is of the form as above. Choose  $M, N \in C_G(J(k,r))$  such that  $R = M^{-1}BN$ , where

$$R = \left( \begin{array}{c|c|c} I' & & \\ \hline & I_{k-2r} & \\ \hline S & & I' \end{array} \right) \text{ with } S = \left( \begin{array}{c|c} & \\ \hline & S' \end{array} \right), I' = \left( \begin{array}{c|c} I_l & \\ \hline & \end{array} \right)$$

for some  $S' \in M(r-l, r-l)$  and  $0 \leq l \leq r$ , thus  $x = \left( \begin{array}{c|c} N & \\ \hline & M \end{array} \right) \left( \begin{array}{c|c} J(k,r) & \\ \hline R & J(k,r) \end{array} \right) \left( \begin{array}{c|c} N & \\ \hline & M \end{array} \right)^{-1}$ . For simplicity, we shall assume that  $M = N = I_k$ . Indeed, the same argument below works for  $y = \left( \begin{array}{c|c} N^{-1}PN & N^{-1}QM \\ \hline M^{-1}PM & \end{array} \right)$ .

Observe that  $x \sim y$  implies that

$$(18) \quad J(k,r)P + PJ(k,r) = RQ = QR \text{ and } PR = RP.$$

Write  $P = (P_i)$  with  $P_1, P_9 \in M(r,r)$ ,  $P_5 \in M(k-2r, k-2r)$ ,  $1 \leq i \leq 9$  as below. Then, by the first equation in (18), we obtain that

$$(19) \quad P_3 = \left( \begin{array}{c|c} P'_3 & \\ \hline & \end{array} \right), Q_1 = \left( \begin{array}{c|c} P'_3 & \\ \hline & Q'_1 \end{array} \right), Q_2 = P_6 = \left( \begin{array}{c|c} P'_6 & \\ \hline & \end{array} \right), Q_4 = \left( \begin{array}{c|c} Q'_4 & \\ \hline & Q''_4 \end{array} \right), Q_5 = P_2 = \left( \begin{array}{c|c} P'_2 & \\ \hline & \end{array} \right),$$

$Q_3 = 0$ , and  $P_1 + P_9 = \left( \begin{array}{c|c} Q'_4 & \\ \hline & Q'_1 S' = S' Q'_1 \end{array} \right)$  for some  $P'_3, Q'_4 \in M(l,l)$ ,  $P'_6 \in M(k-2r, l)$ ,  $P'_2 \in M(l, k-2r)$ . Similarly, by the second equation in (18) we have

$$(20) \quad P_1 = \left( \begin{array}{c|c} P'_1 & \\ \hline & P''_1 \end{array} \right), P_4 = \left( \begin{array}{c|c} P'_4 & \\ \hline & \end{array} \right), P_7 = \left( \begin{array}{c|c} P'_7 & \\ \hline & P''_7 \end{array} \right), P_8 = \left( \begin{array}{c|c} P'_8 & \\ \hline & \end{array} \right), P_9 = \left( \begin{array}{c|c} P'_9 & \\ \hline & P''_9 \end{array} \right),$$

where  $P'_1, P'_7, P'_9 \in M(l,l)$ ,  $P'_4 \in M(k-2r, l)$ , and  $P'_8 \in M(l, k-2r)$  such that  $(P'_1)''^2 = (P'_9)''^2 = I_{r-l}$ .

Using (19) and (20) together with elementary operations we see that  $y$  is equivalent to  $y' = \left( \begin{array}{c|c} P & Q' \\ \hline & P \end{array} \right)$ , i.e.  $\text{rank } y = \text{rank } y'$  where

$$P = \left( \begin{array}{c|c|c|c|c} P'_1 & & P'_2 & P'_3 & \\ \hline & P''_1 & & & \\ \hline P'_4 & & P'_5 & P'_6 & \\ \hline P'_7 & & P'_8 & P'_9 & \\ \hline & P''_7 & & & P''_9 \end{array} \right) \text{ and } Q' = \left( \begin{array}{c|c|c|c|c} & & & & \\ \hline & Q'_1 & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & Q'_4 + Q'_1 S' & & & Q'_1 \end{array} \right).$$

Let  $P'$  be the matrix obtained by deleting the second and fifth rows and columns of  $P$ . Then, we have  $(P')^2 = I_{k+2l-2r}$  as  $(P)^2 = I_k$ , thus  $\text{rank}(P' + I_{k+2l-2r}) \leq (k+2l-2r-1)/2$ . Let  $y''$  be the matrix obtained from  $y'$  by deleting all rows and columns of  $P$  and  $Q'$  except the second and fifth rows and columns in each  $P$  and  $Q'$ . Then, it follows by  $(P)^2 = I_k$  and  $PQ = QP$  that  $(y'')^2 = I_{4r-4l}$ . Hence,  $\text{rank}(y'' + I_{4r-4l}) \leq 2r-2l$ . Therefore,  $\text{rank}(y + I_n) = \text{rank}(y' + I_n) = 2(\text{rank}(P' + I_{k+2l-2r}) + \text{rank}(y'' + I_{4r-4l})) \leq k-1$ , which implies that  $y \notin X_k$ , i.e.,  $d(t, x) \geq 4$ .  $\square$

In the remaining cases, we gives lower bounds of the diameter in  $\text{char}(F) = 2$ .

**Proposition 2.8.** *Let  $F$  be a field of characteristic 2 and  $n \geq 3$ . Then,  $\text{Diam } \Gamma(G, X_k) \geq 4$  if  $n = 2k$  with  $k$  odd and  $\text{Diam } \Gamma(G, X_k) \geq 3$  otherwise.*

*Proof.* By Lemma 2.7, it remains to prove the case except that  $n = 2k$  with  $k$  odd. We find an involution  $x \in X_k$  which does not commute with any involution in  $\Delta_1(t)$ . Observe that any square matrix which commutes with  $J(k, r)$  is of the form  $\left( \begin{array}{c|c|c} X & & \\ \hline & Y & \\ \hline & & X \end{array} \right)$  with  $X \in M(r, r)$ . Choose an invertible matrix  $B$  such that any conjugate of  $B$  does not commute with  $J(k, r)$  for all  $1 \leq r \leq [k/2]$ . Such a matrix exists as we can take a diagonal matrix with distinct entries for  $B$  if  $F$  is infinite, and a matrix having an irreducible minimal polynomial of degree  $k$  for  $B$  otherwise. Consider an involution

$$x = \left( \begin{array}{c|c|c} I_{n-2k} & & \\ \hline A & I_k & B \\ \hline & & I_k \end{array} \right) \in X_k \text{ with } A = \begin{cases} 0 & \text{if } 2k \leq n < 3k \\ (B \mid ) & \text{if } 3k \leq n < 4k. \end{cases}$$

We claim that  $d(t, x) \geq 3$ . If not, there exists

$$y = \left( \begin{array}{c|c|c} C & & \\ \hline & D & \\ \hline E & & D \end{array} \right) \in \Delta_1(t) \text{ with } D \in M(k, k), E = 0 \text{ if } 2k \leq n < 3k$$

such that  $AC + DA = BE$  and  $DB = BD$ . Hence, due to the choice of  $B$ , it suffices to show that  $D$  is an involution. It is obvious that  $y \notin X_k$  if  $2k \leq n < 3k$  and  $D = I_k$ , thus  $D$  is an involution. Now assume that  $3k \leq n < 4k$  and  $D = I_k$ . Then,  $C$  is an involution as  $B \in \text{GL}_k(F)$  and  $y \in X_k$ . Hence,  $C = PJ(n-2k, s)P^{-1}$  for some  $1 \leq s < k$  and  $P \in \text{GL}_{n-2k}(F)$ . Write  $P = \left( \begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array} \right)$  with  $P_1 \in M(k, k), P_2 \in M(k, n-3k)$ .

Then, we obtain

$$y = g \left( \begin{array}{c|c|c} J(n-2k, s) & & \\ \hline & I_k & \\ \hline (P_1 \mid P_2)(J(n-2k, s) + I_{n-2k}) & & I_k \end{array} \right) g^{-1} \text{ with } g = \left( \begin{array}{c|c|c} P & & \\ \hline & I_k & \\ \hline & & I_k \end{array} \right)$$

and the rank of  $(P_1 \mid P_2)(J(n-2k, s) + I_{n-2k})$  is at most  $s$ . Since the nonzero elements are in the first  $s$ -columns,  $\text{rank } y = s < k$ , concluding that  $y \notin X_k$ . Thus,  $D$  is an involution.  $\square$

Now we establish tight upper bounds of the diameter in case  $\text{char}(F) = 2$ . For the case of  $3k \leq n$ , sharp upper bounds were obtained in [6, Theorem 1.3]. In the following theorem, we consider the remaining case where  $2k \leq n < 3k$ . In the proof, we shall use the involution in (3) and Proposition 2.2.

**Theorem 2.9.** *Let  $F$  be a field of characteristic 2 and  $n \geq 3$ . Then,  $\text{Diam } \Gamma(G, X_k) = 4$  if  $n = 2k$  with  $k$  odd and  $\text{Diam } \Gamma(G, X_k) = 3$  otherwise.*

*Proof.* By Proposition 2.8 it suffices to show that the lower bound is sharp. Let  $x \in X_k$ ,  $\dim([V, t] \cap [V, x]) = m$ , and  $s = 3k - n$ . For the case of  $3k \leq n$ , we refer to the proof of [6, Theorem 1.3]. So we may assume that  $2k \leq n < 3k$ . First, we consider the case where  $2m \geq k$ . Then, by transitivity we have  $[V, x] = [V, gt_m g^{-1}]$  for some  $g \in C_G([V, t])$ . Hence, by Lemma 2.1, we obtain  $d(x, gt_m g^{-1}) = d(t, gt g^{-1}) = 1$ . As  $t_m \sim t$ , we conclude that  $d(t, x) \leq d(t, gt g^{-1}) + d(gt g^{-1}, gt_m g^{-1}) + d(gt_m g^{-1}, x) \leq 3$ .

Now we assume that  $2m < k$ . Again by transitivity, there exists a block lower triangular matrix  $g$  such that  $[V, x] = [V, gt_m g^{-1}] = [V, g(w_{k-m} t^T w_{k-m}) g^{-1}] = [V, gw_{k-m} w_0 t w_0 w_{k-m} g^{-1}]$ . Hence, we have

$$(21) \quad x = gw_{k-m} w_0 \left( \begin{array}{c|c} I_{n-k} & \\ \hline B & I_k \end{array} \right) w_0 w_{k-m} g^{-1}$$

for some  $B \in M(k, n-k)$  with  $\text{rank}(B) = k$ . Let  $y$  denote the middle matrix in the above decomposition. We divide the proof into the following three cases (For the second and third cases, we always exclude the first case).

Case:  $n = 2k$  with  $k$  odd. Let  $k = 2r + 1$ . Write  $w_0 y w_0 = u t^T u^{-1}$  for some block diagonal matrix  $u \in C_G([V, t])$ . If  $m = 0$ , then by (21) we have  $x = h t^T h^{-1}$ , where  $h = gu$ . In this case, one has the following

commuting involutions in  $X_k$

$$(22) \quad t^T \sim \left( \frac{J(k, r)^T}{J(k, r)^T} \middle| \frac{e_{(r+1)(r+1)}}{J(k, r)^T} \right) \sim \left( \frac{I(k, r)^T}{e_{1k}} \middle| \frac{I'(k, r)^T}{I'(k, r)^T} \right) =: z$$

where,  $I'(k, r) = \left( \frac{I_r}{I_r} \middle| I_{k-r} \right)$ . Since  $hzh^{-1}$  is block lower triangular, it follows from Proposition 2.2 that  $d(t, hzh^{-1}) \leq 2$ , thus  $d(t, x) \leq 4$ . Similarly, if  $m > 0$ , then instead of (22) we use

$$t^T \sim \left( \frac{I(k, r)}{I(k, r)} \middle| \frac{e_{11}}{I(k, r)} \right) =: z'.$$

As  $w_{k-m}z'w_{k-m}$  is block lower triangular, it follows from Proposition 2.2 that  $d(t, gw_{k-m}z'w_{k-m}g^{-1}) \leq 2$ , thus  $d(t, x) \leq 4$ .

**Case:  $2k \leq n < 3k$  with  $m = 0$ .** It follows from (21) that  $x = gw_0yw_0g^{-1}$ . Let  $P \in \text{GL}_k(F)$  and let  $Q \in \text{GL}_{n-k}(F)$  be a block lower triangular matrix of the form  $Q = \left( \frac{Q_1}{Q_2} \middle| Q_3 \right)$ ,  $Q_1 \in M(n-2k, n-2k)$ ,  $Q_3 \in M(k, k)$  such that  $PBQ = \left( D_1 \middle| D_2 \right)$ , where

$$D_1 = \left( \frac{I_l}{I_l} \middle| \right) \in M(k, n-2k) \text{ and } D_2 = \left( \middle| \frac{I_{k-l}}{I_{k-l}} \right) \in M(k, k)$$

for some  $0 \leq l \leq n-2k$ . Then, we obtain

$$(23) \quad x = h \left( \frac{I_{n-2k}}{D_1} \middle| \frac{I_k}{I_k} \middle| \frac{D_2}{I_k} \right) h^{-1} \text{ with } h = gw_0 \left( \frac{Q}{P^{-1}} \right) w_0 \in C_G([V, t]).$$

Let  $y'$  denote the middle matrix in the above decomposition. First assume that either  $l < n-2k$  or  $l = n-2k$  with  $s$  even. Consider an involution

$$z = \left( \frac{I_{n-2k}}{I_{n-2k}} \middle| \frac{I(k, \frac{s}{2})^T}{I(k, \frac{s}{2})^T} \right) \text{ (resp. } \left( \frac{I_{n-2k}}{I_{n-2k-1}} \middle| \frac{I(k, \lceil \frac{s}{2} \rceil)^T}{I(k, \lceil \frac{s}{2} \rceil)^T} \right) \in \Delta_1(t)$$

if  $s$  even (resp. otherwise). Then, by a direct computation we have  $z \sim y'$ . Hence, by Proposition 2.2 we conclude that  $d(t, x) = d(t, hzh^{-1}) + d(hzh^{-1}, x) \leq 3$ . If  $l = n-2k$  with  $s$  odd, the same proof works if we replace  $z$  by

$$z' = \left( \frac{I_{n-2k}}{I_{n-2k}} \middle| \frac{I'(k, \lceil \frac{s}{2} \rceil)}{I(k, \lceil \frac{s}{2} \rceil)^T} \right) \in \Delta_1(t), \text{ where } I'(k, \lceil \frac{s}{2} \rceil) = \left( \frac{I_{k-\lceil \frac{s}{2} \rceil}}{1} \middle| \frac{I_{\lceil \frac{s}{2} \rceil}}{I_{\lceil \frac{s}{2} \rceil}} \right).$$

**Case:  $2k \leq n < 3k$  with  $m > 0$ .** First assume that  $s \leq 2m$ . Then,  $z := w_{k-m}w_0J(n, k)w_0w_{k-m}$  is block lower triangular. Hence, by Proposition 2.2 we have  $d(t, gzg^{-1}) \leq 2$ . As  $J(n, k) \sim y$ , we obtain  $d(t, x) \leq 3$ . From now on we assume that  $s > 2m$ .

Let  $P \in \text{GL}_k(F)$  and let  $Q \in \text{GL}_{n-k}(F)$  be block lower triangular matrices of the forms

$$(24) \quad P = \left( \frac{P_1}{P_2} \middle| \frac{P_3}{P_3} \right) \text{ and } Q = \left( \frac{Q_1}{Q_2} \middle| \frac{Q_3}{Q_4} \middle| \frac{Q_5}{Q_5} \right), P_1 \in \text{GL}_{k-m}(F), Q_1 \in \text{GL}_{n-2k}(F), Q_3 \in \text{GL}_m(F)$$

such that  $PBQ = \left( \begin{array}{c|c|c} D_1 & D_2 & \\ \hline & & D_3 \\ \hline & D_4 & \end{array} \right)$ , where

$D_1 = \left( \begin{array}{c|c} I_{l'} & \\ \hline & \end{array} \right) \in M(k-m-l, n-2k)$ ,  $D_2 = \left( \begin{array}{c|c} & \\ \hline I_{l''} & \end{array} \right) \in M(k-m-l, m)$ ,  $D_3 = \left( \begin{array}{c|c} & \\ \hline & I_l \end{array} \right) \in M(k-m, k-m)$  for some  $s-2m \leq l \leq k-m$  and  $l' + l'' = k-m-l$ . Then,  $x = ghw_{k-m}w_0y'(ghw_{k-m}w_0)^{-1}$ , where  $h = w_{k-m}w_0 \left( \begin{array}{c|c} Q & \\ \hline & P^{-1} \end{array} \right) w_0w_{k-m}$  is a block lower triangular matrix in  $C_G([V, t])$  and  $y' = \left( \begin{array}{c|c} I_{n-k} & \\ \hline PBQ & I_k \end{array} \right)$ .

Assume that either  $l > s-2m$  or  $l = s-2m$  with  $s$  even. By multiplying a permutation  $P' \in \text{GL}_k(F)$ ,  $Q' \in \text{GL}_{n-k}(F)$  of the form in (24) we change  $D_3$  in  $PBQ$  to

$$(25) \quad \left( \begin{array}{c|c|c|c} I_p & & & \\ \hline & I_{\lceil \frac{s}{2} \rceil - m} & & \\ \hline & & D'_3 & \\ \hline & & & I_{\lceil \frac{s}{2} \rceil - m} \end{array} \right) \text{ with } p = \begin{cases} l + 2m - 2\lceil \frac{s}{2} \rceil & \text{if } l \leq k - 2m \\ l + m - 2\lceil \frac{s}{2} \rceil & \text{if } l > k - 2m \text{ and } l \geq 2\lceil \frac{s}{2} \rceil - m \\ 0 & \text{if } l > k - 2m \text{ and } l < 2\lceil \frac{s}{2} \rceil - m, \end{cases}$$

where

$$D'_3 = \begin{cases} 0 & \text{if } l \leq k - 2m \\ I_m & \text{if } l > k - 2m \text{ and } l \geq 2\lceil \frac{s}{2} \rceil - m \\ \left( \begin{array}{c|c} I_{l-2(\lceil \frac{s}{2} \rceil - m)} & \\ \hline & \end{array} \right) & \text{if } l > k - 2m \text{ and } l < 2\lceil \frac{s}{2} \rceil - m. \end{cases}$$

We replace  $P$  by  $P'P$  and  $Q$  by  $QQ'$  in both  $h$  and  $y'$ , which with abuse of notations we still denote by  $h$  and  $y'$ . Consider an involution

$$z' = \left( \begin{array}{c|c|c} I_{n-2k} & & \\ \hline & I_1(k, \lceil \frac{s}{2} \rceil) & \\ \hline I_{n-2k+s-2\lceil \frac{s}{2} \rceil} & & \\ \hline & I_{2m} & I_2(k, \lceil \frac{s}{2} \rceil) \end{array} \right),$$

$$\text{where } I_1(k, \lceil \frac{s}{2} \rceil) = \left( \begin{array}{c|c|c} I_{k-s} & & \\ \hline & I_{\lceil \frac{s}{2} \rceil} & \\ \hline & & I_{\lceil \frac{s}{2} \rceil - m} \\ \hline & & I_{\lceil \frac{s}{2} \rceil} \end{array} \right) \text{ and } I_2(k, \lceil \frac{s}{2} \rceil) = \left( \begin{array}{c|c|c} I_{k-s} & & \\ \hline & I_{\lceil \frac{s}{2} \rceil} & I_{\lceil \frac{s}{2} \rceil - m} \\ \hline & & I_{\lceil \frac{s}{2} \rceil} \end{array} \right).$$

Then, by a direct calculation, we see that  $y' \sim z'$ . Since  $ghw_{k-m}w_0z'(ghw_{k-m}w_0)^{-1}$  is block lower triangular in  $C_G[V, t]$ , its distance from  $t$  is at most two by Proposition 2.2. Therefore,  $d(t, x) \leq 3$ .

Similarly, if  $l = s-2m$  with  $s$  odd, then the same proof works if we replace (25) and  $z'$  by

$$\left( \begin{array}{c|c|c|c} & & & \\ \hline & I_{\lceil \frac{s}{2} \rceil - m} & & \\ \hline & & & \\ \hline & & 1 & \\ \hline & & & I_{\lceil \frac{s}{2} \rceil - m} \end{array} \right) \text{ and } \left( \begin{array}{c|c|c} I_{n-2k} & & \\ \hline & I_1(k, \lceil \frac{s}{2} \rceil) & \\ \hline I_{n-2k} & & \\ \hline & I_{2m} & I'_2(k, \lceil \frac{s}{2} \rceil) \end{array} \right)$$

$$\text{where the middle zero block in the first one has size } m \text{ and } I'_2(k, \lceil \frac{s}{2} \rceil) := \left( \begin{array}{c|c|c} I_{k-s} & & \\ \hline & I_{\lceil \frac{s}{2} \rceil} & I_{\lceil \frac{s}{2} \rceil - m} \\ \hline & & 1 \\ \hline & & I_{\lceil \frac{s}{2} \rceil} \end{array} \right),$$

respectively.  $\square$

Applying the same argument as used in the proof of Theorem 2.9, we determine the diameter of the commuting graph on the set of all involutions.

**Corollary 2.10.** *Let  $F$  be a field of characteristic 2,  $n \geq 3$ , and  $X$  the set of all involutions in  $G$ . Then,  $\text{Diam } \Gamma(G, X) = 3$ .*

*Proof.* If  $n = 3$ , then  $X = X_1$ , thus the result follows from [6, Theorem 3.1]. So we assume that  $n \geq 4$ . Let  $x \in X_k$  for some  $1 \leq k \leq n/2$ . In order to establish our upper bound, it suffices to show that  $d(x, t'_i) \leq 3$  for all  $1 \leq i \leq n/2$ , where  $t'_i = J(n, i) \in X_i$ . Note that  $t'_i \sim t'_k$  for all  $i, k$ . If  $n \geq 4k$ , then by [6, Lemma 4.3], we have  $d(x, t'_k) \leq 2$ , thus  $d(x, t'_i) = d(x, t'_k) + d(t'_k, t'_i) \leq 3$ . Hence, we assume that  $n/4 < k \leq n/2$ . It follows from the proof of Theorem 2.9 that  $x$  is either of the form (21) for  $2m < k$  or

$$(26) \quad x = gw_m \left( \begin{array}{c|c} I_{n-k} & \\ \hline B & I_k \end{array} \right) w_m g^{-1}$$

for  $2m \geq k$ . Observe that the middle matrices in (21) and (26) commute with  $t'_1$ . Hence, it is enough to show that  $d(x_1, t'_i) \leq 2$  for any  $x_1 \in X_1$  and all  $2 \leq i \leq n/2$ . Let  $m_1 = \dim([V, x_1] \cap [V, t_1])$ . If  $m_1 = 1$ , then by (26) we have a path  $x_1 \sim t'_1 \sim t'_i$ . Otherwise, we have either  $x_1 = gw_0 t'_1 w_0 g^{-1}$  or  $x_1 = gw_0 t_1 w_0 g^{-1}$  for some  $g \in C_G([V, t_1])$ , i.e.,

$$(27) \quad x_1 = g \left( \begin{array}{c|c} J(n-1, 1) & \\ \hline & 1 \end{array} \right) g^{-1} \text{ or } x_1 = g \left( \begin{array}{c|c} I_{n-1} & \overline{1} \\ \hline & 1 \end{array} \right) g^{-1} \text{ with } g = \left( \begin{array}{c|c} P & \\ \hline Q & r \end{array} \right)$$

for some  $P \in \text{GL}_{n-1}(F)$ ,  $r \in F^\times$ .

We shall find  $y_1 \in X_1$  such that  $x_1 \sim y_1 \in C_G(t'_i)$  for each case in (27). Consider the first case in (27). Write  $P^{-1} = \left( \begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array} \right)$  with  $P_2 \in M(i-1, i-1)$  and  $P_4 \in M(n-i, i-1)$ . Let  $M \in \text{GL}_{n-1}(F)$  and  $N \in \text{GL}_{i-1}(F)$  such that

$$(28) \quad \left( \begin{array}{c|c} P_2 & \\ \hline P_4 & \end{array} \right) = M \left( \begin{array}{c|c} I_{i-1} & \\ \hline & \end{array} \right) N.$$

Find a nonzero row  $B \in M(1, n-1)$  such that the first  $i-1$  elements are all zero and the last element of  $BM^{-1}$  is zero. Then, by (28) the last  $i-1$  elements of  $BM^{-1}P^{-1}$  are all zero. Take

$$y_1 = g \left( \begin{array}{c|c} I_{n-1} & \\ \hline BM^{-1} & 1 \end{array} \right) g^{-1} = \left( \begin{array}{c|c} I_{n-1} & \\ \hline rBM^{-1}P^{-1} & 1 \end{array} \right).$$

By construction, this involution is contained in  $C_G(t'_i)$  and commutes with  $x_1$ .

Now we consider the second involution in (27). By the argument given in the proof of Theorem 2.9, we can choose

$$M = \left( \begin{array}{c|c} M_1 & \\ \hline & I_{n-i-1} \end{array} \right), N = \left( \begin{array}{c|c} N_1 & \\ \hline N_2 & N_3 \end{array} \right) \in \text{GL}_{n-1}(F), N_1 \in \text{GL}_i(F)$$

such that

$$P' := M^{-1}PN^{-1} = \left( \begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & \end{array} \right) \text{ with } P_1 = \left( \begin{array}{c|c} I_l & \\ \hline & \end{array} \right) \in M(i, i), P_2 = \left( \begin{array}{c|c} & \\ \hline & I_{i-l} \end{array} \right) \in M(i, n-1-i)$$

for some  $0 \leq l \leq i$ . First assume  $l > 0$ . As  $i < n-1$ , there is a zero column in  $\left( \begin{array}{c|c} P_1 & P_2 \end{array} \right)$ , say  $j$ -th column. Take

$$(29) \quad y_1 = g \left( \begin{array}{c|c} N^{-1}(I_{n-1} + e_{j1})N & \\ \hline & 1 \end{array} \right) g^{-1} = \left( \begin{array}{c|c} M & \\ \hline & 1 \end{array} \right) \left( \begin{array}{c|c} P'(I_{n-1} + e_{j1})P'^{-1} & \\ \hline QN^{-1}e_{j1}P'^{-1} & 1 \end{array} \right) \left( \begin{array}{c|c} M^{-1} & \\ \hline & 1 \end{array} \right).$$

Then, one can easily check that the second middle matrix in (29) is contained in  $C_G(t'_i)$  and the first middle matrix in (29) commutes with the middle matrix in (27). Therefore,  $d(x_1, t'_i) \leq 2$ , thus  $d(x, t'_i) \leq 3$ . If  $l = 0$ , then  $i \leq n-1-i$  and  $\left( \begin{array}{c|c} P_1 & P_2 \end{array} \right) = \left( \begin{array}{c|c} & \\ \hline & I_i \end{array} \right)$ . Similarly, we can choose

$$M' = \left( \begin{array}{c|c|c} I_i & & \\ \hline M_2 & M_3 & \\ & M_4 & I_{i-1} \end{array} \right), N' = \left( \begin{array}{c|c|c} N_4 & & \\ \hline N_5 & N_6 & \\ & & I_i \end{array} \right) \in \text{GL}_n(F), M_3 \in \text{GL}_{n-2i}(F), N_4 \in \text{GL}_i(F)$$



such that

$$P'' := M'^{-1}P'N'^{-1} = \left( \begin{array}{c|c|c} & & I_i \\ \hline \frac{P'_1}{P'_3} & \frac{P'_2}{P'_3} & \\ \hline \end{array} \right) \text{ with } P'_1 = \left( \begin{array}{c|c} I_s & \\ \hline \end{array} \right) \in M(n-2i, i), P'_2 = \left( \begin{array}{c|c} & \\ \hline I_{n-2i-s} \end{array} \right)$$

for some  $0 < s \leq n-2i$  ( $s \neq 0$  as  $P'_2 \in M(n-2i, n-2i-1)$ ). Observe that the first entries of the last  $i-1$  columns of  $\left( \frac{P'_1}{P'_3} \middle| \frac{P'_2}{P'_3} \right)^{-1}$  are all zero. By replacing  $M, N, P'$  in (29) by  $MM', N'N, P''$ , respectively, we get the same result as in the previous case  $l > 0$ .

To prove the lower bound, consider the following involutions

$$x = \left( \begin{array}{c|c} I_k & B \\ \hline & I_k \end{array} \right) \text{ if } n = 2k, \text{ and } x = \left( \begin{array}{c|c|c} I_k & & B \\ \hline & 1 & \\ \hline & & I_k \end{array} \right) \text{ if } n = 2k+1,$$

where  $B$  is an invertible matrix that does not commute with any  $J(k, r)$  for all  $1 \leq r \leq k/2$  as in Proposition 2.8. We show that  $d(x, t'_k) \geq 3$ . Suppose that there exists an involution  $y$  such that  $x \sim y$ . Then,  $y$  has the following forms

$$y = \left( \begin{array}{c|c} A & C \\ \hline & B^{-1}AB \end{array} \right) \text{ and } y = \left( \begin{array}{c|c|c} A & & C \\ \hline & d & E \\ \hline & & B^{-1}AB \end{array} \right).$$

for some  $A \in \text{GL}_k(F)$  and  $d \in F^\times$  in the corresponding cases. If  $y \in C_G(t'_k)$ , then  $C = E = 0$  and  $BA = AB$  in both cases, which implies  $y$  is the identity. This establishes the lower bound.  $\square$

### 3. FOUR-DIMENSIONAL LINEAR GROUPS

In the present section, we determine the structure of  $\text{GL}_4(F)$  over a finite field in Propositions 3.1 and 3.5. In particular, by applying the results in the previous section we provide the number of involutions  $\Delta_i(t)$  for each distance  $i$ .

**Proposition 3.1.** *Let  $G = \text{GL}_4(F)$  over a finite field  $F$  with  $q$  elements and let  $X_k$  be the class of involutions for  $k = 1, 3$  in case  $\text{char}(F) \neq 2$  and  $k = 1$  in case  $\text{char}(F) = 2$ . Then, for any such  $k$ ,  $\text{Diam } \Gamma(G, X_k) = 2$ . Moreover,*

$$|\Delta_1(t)| = \begin{cases} q^4 + 2q^3 - q^2 - q - 2 & \text{if } \text{char}(F) = 2, \\ q^4 + q^3 + q^2 & \text{if } \text{char}(F) \neq 2. \end{cases}$$

$$|\Delta_2(t)| = \begin{cases} q^6 + q^5 - 2q^3 & \text{if } \text{char}(F) = 2, \\ q^6 + q^5 - q^2 - 1 & \text{if } \text{char}(F) \neq 2. \end{cases}$$

*Proof.* It follows from (1) that

$$|X_1| = \begin{cases} |\text{GL}_4(F)|/q^5 |\text{GL}_2(F)| |\text{GL}_1(F)| = (q^4 - 1)(q^2 + q + 1) & \text{if } \text{char}(F) = 2, \\ |\text{GL}_4(F)|/|\text{GL}_3(F)| |\text{GL}_1(F)| = q^3(q^2 + 1)(q + 1) & \text{if } \text{char}(F) \neq 2. \end{cases}$$

By Theorem 1.1 (i) and a symmetric argument, it suffices to compute  $|\Delta_1(t)|$  for  $k = 1$ . If  $\text{char}(F) \neq 2$ , then by (1) we see that  $\Delta_1(t)$  consists of involutions  $\left( \begin{array}{c|c} A & \\ \hline & 1 \end{array} \right)$  such that  $A$  is an involution in  $\text{GL}_3(F)$  with  $k = 1$ . Hence,  $|\Delta_1(t)| = q^4 + q^3 + q^2$ .

isume that  $\text{char}(F) = 2$ . Then, it follows by (1) that  $\Delta_1(t)$  consists of the following involutions

$$\left( \begin{array}{c|c|c} A & B & \\ \hline & 1 & \\ \hline C & d & 1 \end{array} \right)$$

such that  $A^2 = I_2, AB = B, CA = C, CB = 0$ . If  $A = I_2$ , then by using  $\text{rank}(I_4 + x) = 1$  one can easily check that the possible number of  $x$  is  $2q^3 - q - 2$ . Now assume that  $A = PI(2, 1)P^{-1}$  for some  $P \in \text{GL}_2(F)$ . Then, the involution  $x$  is decomposed as

$$(30) \quad \left( \begin{array}{c|c} P & \\ \hline & I_2 \end{array} \right) \left( \begin{array}{cc|c} 1 & & \\ \hline 1 & 1 & b \\ c & & d \quad 1 \end{array} \right) \left( \begin{array}{c|c} P^{-1} & \\ \hline & I_2 \end{array} \right)$$

for some  $b, c \in F$  with  $bc = d$ . If  $b = c = 0$ , then the possible number of  $x$  is  $q^2 - 1$ . If either  $b = 0$  and  $c \neq 0$  or  $b \neq 0$  and  $c = 0$ , then the number of  $\left( \begin{array}{c|c} P & \\ \hline & I_2 \end{array} \right)$ -conjugacy class of the middle involution in (30) is  $(q^2 - 1)(q - 1)$ , so is the possible number of  $x$  in each case. Similarly, if  $b \neq 0$  and  $c \neq 0$ , then the possible number of  $x$  is  $(q^2 - 1)(q - 1)^2$ , thus

$$|\Delta_1(t)| = q^4 + 2q^3 - q^2 - q - 2,$$

which completes the proof.  $\square$

Now we consider the conjugacy class  $X_2$ . By the same argument as in the proof of Proposition 3.1, we have

$$(31) \quad |X_2| = \begin{cases} q(q^4 - 1)(q^3 - 1) & \text{if } \text{char}(F) = 2, \\ q^4(q^2 + 1)(q^2 + q + 1) & \text{if } \text{char}(F) \neq 2. \end{cases}$$

**Lemma 3.2.** *Let  $U_i = \{x \in X_2 \mid \dim([V, t] \cap [V, x]) = i\}$  for  $0 \leq i \leq 2$ . Then,*

$$\begin{aligned} |U_0| &= q^4(q^2 - 1)(q^2 - q), & |U_1| &= (q^2 - 1)^2(q^2 + q^3), & |U_2| &= (q^2 - 1)(q^2 - q) - 1 & \text{if } \text{char}(F) = 2, \\ |U_0| &= q^8, & |U_1| &= q^5(q + 1)^2, & |U_2| &= q^4 - 1 & \text{if } \text{char}(F) \neq 2. \end{aligned}$$

*Proof.* The result for  $U_2$  immediately follows from Lemma 2.1. By (31), it suffices to compute  $|U_0|$ . We write  $g \in C_G([V, t])$  as

$$(32) \quad g = \left( \begin{array}{c|c} P & \\ \hline & Q \end{array} \right) \left( \begin{array}{c|c} I_2 & \\ \hline R & I_2 \end{array} \right) \text{ with } P, Q \in \text{GL}_2(F), R \in \text{M}(2, 2).$$

We denote by  $h$  the block diagonal matrix in (32). Then, by the proof of Theorem 2.5 and (21) we have

$$x = \begin{cases} g \left( \begin{array}{c|c} I_2 & B \\ \hline & I_2 \end{array} \right) g^{-1} = h \left( \begin{array}{cc|c} I_2 + BR & & B \\ \hline RBR & & I_2 + RB \end{array} \right) h^{-1} & \text{if } \text{char}(F) = 2, \\ g \left( \begin{array}{c|c} -I_2 & B \\ \hline & I_2 \end{array} \right) g^{-1} = h \left( \begin{array}{cc|c} -I_2 - BR & & B \\ \hline -2R - RBR & & I_2 + RB \end{array} \right) h^{-1} & \text{if } \text{char}(F) \neq 2, \end{cases}$$

where  $B$  is invertible in case  $\text{char}(F) = 2$ . Replacing  $B$  and  $R$  by  $PBQ^{-1}$  and  $QRP^{-1}$ , respectively, we may assume that  $h = I_4$ . Hence, the possible number of  $x$  is the product of possible numbers of  $B$  and  $R$ , thus the result follows.  $\square$

By Lemma 2.1, we see that  $|U_2| = |\Delta_1(t) \cap U_2|$  if  $\text{char}(F) = 2$  and  $|U_2| = |\Delta_2(t) \cap U_2|$  otherwise. Hence, it remains to analyze the structures of  $U_1$  and  $U_0$ .

**Lemma 3.3.** *If  $\text{char}(F) = 2$ , then*

$$|\Delta_1(t) \cap U_1| = q^2(q^2 - 1), \quad |\Delta_2(t) \cap U_1| = q^2(q^2 - 1)(2q^2 - q - 2), \quad |\Delta_3(t) \cap U_1| = q^4(q^3 - q^2 - q + 1).$$

*Otherwise, we have*

$$|\Delta_1(t) \cap U_1| = q^2(q + 1)^2, \quad |\Delta_2(t) \cap U_1| = q(q + 1)^2(q - 1)(q^3 + q + 1), \quad |\Delta_3(t) \cap U_1| = q(q + 1)^3(q - 1)^2.$$

*Proof.* We first assume that  $\text{char}(F) = 2$ . Let  $x \in U_1$  and  $H = C_G([V, t])$ . Then, it follows from the proof of Theorem 2.9 and Lemma 2.1 that  $x = hw_1 \left( \frac{I_2}{B} \middle| \frac{I_2}{I_2} \right) w_1 h^{-1}$  for some  $h \in H$  and  $B \in \text{GL}_2(F)$ . If  $B$  is lower triangular, then one can find  $h' \in H$  such that  $x = h'w_1tw_1h'^{-1}$ . Otherwise, one finds  $h'' \in H$  such that  $x = h''w_1 \left( \frac{I_2}{B'} \middle| \frac{I_2}{I_2} \right) w_1 h''^{-1}$ , where  $B' = \left( \frac{1}{1} \middle| \frac{1}{1} \right)$ . Hence,  $U_1 = Hy_1H^{-1} \cup Hy_2H^{-1}$ , where

$$y_1 = \left( \frac{I(2, 1)}{I(2, 1)} \middle| \frac{I(2, 1)}{I(2, 1)} \right) \text{ and } y_2 = \left( \frac{I_2}{e_{21}} \middle| \frac{e_{21}}{I_2} \right).$$

By a simple calculation, we have  $|Hy_1H^{-1}| = q^2(q^2 - 1)^2$  and  $|Hy_2H^{-1}| = q^3(q^2 - 1)^2$ .

Let  $x \in Hy_1H^{-1}$ . Then, by Proposition 2.2,  $x \in \Delta_1(t) \cup \Delta_2(t)$ , thus it suffices to compute  $|\Delta_1(t) \cap Hy_1H^{-1}|$ . Let  $x = hy_1h^{-1} \in \Delta_1(t) \cap Hy_1H^{-1}$  with  $h = \left( \frac{P}{R} \middle| \frac{Q}{Q} \right) \in H$ . Then, by (1)  $PQ^{-1} \in C_{\text{GL}_2(F)}(I(2, 1))$ , thus we see that  $x \in Ky_1K^{-1}$ , where  $K = \left\{ \left( \frac{P}{R} \middle| \frac{P}{P} \right) \mid P \in \text{GL}_2(F), R \in \text{M}(2, 2) \right\}$ . Therefore, we obtain

$$(33) \quad |\Delta_1(t) \cap Hy_1H^{-1}| = q^2(q^2 - 1), \text{ thus } |\Delta_2(t) \cap Hy_1H^{-1}| = q^2(q^4 - 3q^2 + 2).$$

Let  $x \in Hy_2H^{-1}$ . Then, by (1) we see that  $x \in \Delta_2(t) \cup \Delta_3(t)$ , thus it suffices to calculate  $|\Delta_2(t) \cap Hy_2H^{-1}|$ . If  $x \in \Delta_2(t) \cap Hy_2H^{-1}$ , then there exists  $z = h \left( \frac{E_{21}(a)}{be_{21}} \middle| \frac{E_{21}(c)}{E_{21}(c)} \right) h^{-1} \in \Delta_1(t)$  for some  $a, c \in F^\times, b \in F$  and  $h = \left( \frac{P}{R} \middle| \frac{Q}{Q} \right) \in H$ . Moreover, by (1) we have  $PQ^{-1} = \left( \frac{d}{ce} \middle| \frac{a^{-1}cd}{a^{-1}cd} \right)$  for some  $d \in F^\times$  and  $e \in F$ . Hence, we see that  $x \in Ky_3K^{-1}$ , where  $y_3 = \left( \frac{I_2}{a(cd)^{-1}e_{21}} \middle| \frac{de_{21}}{I_2} \right)$ . Therefore, we obtain that

$$|\Delta_2(t) \cap Hy_2H^{-1}| = q^3(q - 1)^2(q + 1), \text{ thus } |\Delta_3(t) \cap Hy_2H^{-1}| = q^4(q^3 - q^2 - q + 1),$$

which together with (33) completes the proof of case  $\text{char}(F) = 2$ .

Now assume that  $\text{char}(F) \neq 2$ . It is obvious that  $|\Delta_1(t) \cap U_1| = q^2(q + 1)^2$ . Hence, it is enough to calculate  $|\Delta_3(t) \cap U_1|$ . Let  $H'$  be the subgroup of  $H$  consisting of block diagonal matrices and  $x \in U_1$ . Then, by the proof of Theorem 2.5 together with (32),  $x$  is decomposed as

$$(34) \quad \left( \frac{I_2}{R} \middle| \frac{I_2}{I_2} \right) \left( \frac{I(2, 1)}{I(2, 1)} \middle| \frac{be_{21}}{I(2, 1)} \right) \left( \frac{I_2}{-R} \middle| \frac{I_2}{I_2} \right) = \left( \frac{I(2, 1) - be_{21}R}{RI(2, 1) - (I(2, 1) + bRe_{21})R} \middle| \frac{be_{21}}{bRe_{21} + I(2, 1)} \right)$$

for some  $b \in F$  and  $R \in \text{M}(2, 2)$ , up to conjugation in  $H'$ . If  $b = 0$ , then by Proposition 2.2,  $d(t, x) \leq 2$ , thus we may assume that  $b \neq 0$ . Consider the following element  $y \in \Delta_1(t)$

$$y = h \left( \frac{I_2}{R} \middle| \frac{I_2}{I_2} \right) \left( \frac{A}{C} \middle| \frac{B}{D} \right) \left( \frac{I_2}{-R} \middle| \frac{I_2}{I_2} \right) h^{-1}$$

for some  $h \in H'$ ,  $A, B, C, D \in \text{M}(2, 2)$  such that  $x \sim y$ . Then, we obtain that

$$B = 0, C = \left( \frac{c_1}{c_2} \middle| \frac{c_1}{c_2} \right), A = \left( \frac{\pm 1}{bc_1/2} \middle| \frac{\pm 1}{\pm 1} \right), D = \left( \frac{\mp 1}{-bc_2/2} \middle| \frac{\pm 1}{\pm 1} \right)$$

for some  $c_1, c_2 \in F$  and

$$c_1(2 + br_2) = \mp 4r_1, c_2(2 + br_2) = \pm 4r_4, b(c_2r_1 + c_1r_4) = 0 \text{ with } R = \left( \frac{r_1}{r_3} \middle| \frac{r_2}{r_4} \right).$$

Hence, we see that  $x \in \Delta_3(t)$  is equivalent to the conditions  $b \neq 0$ ,  $br_2 = -2$ , and either  $r_1 \neq 0$  or  $r_4 \neq 0$ . Therefore, it follows from (34) that

$$(35) \quad x = \left( \begin{array}{cc|cc} 1 & & & \\ -br_1 & 1 & b & \\ \hline 2r_1 & & -1 & \\ 2r_3 - br_1r_4 & 2r_4 & br_4 & -1 \end{array} \right),$$

up to conjugation in  $H'$  if  $x \in \Delta_3(t)$ . Using (35), we see that  $\Delta_3(t) \cap U_1$  is decomposed into the following union of three orbits

$$H' \left( \begin{array}{cc|cc} 1 & & & \\ 1 & 1 & 1 & \\ \hline & -1 & & \\ 2 & 1 & -1 & \end{array} \right) H'^{-1} \cup H' \left( \begin{array}{cc|cc} 1 & & & \\ -1 & 1 & 1 & \\ \hline 2 & & -1 & \\ & & & -1 \end{array} \right) H'^{-1} \cup H' \left( \begin{array}{cc|cc} 1 & & & \\ -1 & 1 & 1 & \\ \hline 2 & & -1 & \\ & 2 & 1 & -1 \end{array} \right) H'^{-1}$$

with orders  $q(q^2 - 1)^2$ ,  $q(q^2 - 1)^2$ ,  $q(q - 1)(q^2 - 1)^2$ , respectively. Therefore, we obtain  $|\Delta_3(t) \cap U_1| = q(q + 1)^3(q - 1)^2$  and  $|\Delta_2(t) \cap U_1| = q(q + 1)^2(q - 1)(q^3 + q + 1)$ .  $\square$

**Lemma 3.4.** *If  $\text{char}(F) = 2$ , then*

$$|\Delta_1(t) \cap U_0| = 0, \quad |\Delta_2(t) \cap U_0| = q^6(q - 1), \quad |\Delta_3(t) \cap U_0| = q^5(q - 1)(q^2 - q - 1).$$

*Otherwise, we have  $|\Delta_1(t) \cap U_0| = 1$ ,*

$$|\Delta_2(t) \cap U_0| = \frac{1}{2}(q - 1)(q + 1)^2(q^5 - q^3 + 2q^2 + 2), \quad |\Delta_3(t) \cap U_0| = \frac{1}{2}q(q - 1)^2(q + 1)(q^4 + 3q^2 + 2q + 2).$$

*Proof.* First, consider the case  $\text{char}(F) = 2$ . Let  $x \in U_0$ . As in the proof of Lemma 3.4 we may assume that

$$x = \left( \begin{array}{c|c} I_2 + BR & B \\ \hline RBR & I_2 + RB \end{array} \right) = \left( \begin{array}{c|c} I_2 & \\ \hline R & I_2 \end{array} \right) \left( \begin{array}{c|c} I_2 & B \\ \hline & I_2 \end{array} \right) \left( \begin{array}{c|c} I_2 & \\ \hline R & I_2 \end{array} \right),$$

where  $B \in \text{GL}_2(F)$  and  $R \in \text{M}(2, 2)$ . If  $x \in \Delta_1(t)$ , then  $B = 0$ , i.e.,  $x = I_4$ , which implies  $|\Delta_1(t) \cap U_0| = 0$ . We shall compute  $|\Delta_2(t) \cap U_0|$ . Observe that for  $y \in \Delta_1(t)$

$$(36) \quad x \sim y := \left( \begin{array}{c|c} M & \\ \hline N & M \end{array} \right) \Leftrightarrow MB = BM, N = RM + MR.$$

As  $M^2 = I_2$  and  $y$  is an involution, it follows from the first equation in (36) that

$$(37) \quad M = gI(2, 1)g^{-1} \text{ and } B = agE_{21}(b)g^{-1}$$

for some  $g \in \text{GL}_2(F)$ ,  $a \in F^\times$ , and  $b \in F$ . Moreover, for any such  $B$  and  $M$  and any  $R = g \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} g^{-1} \in$

$\text{M}(2, 2)$ , there exists  $N := g \begin{pmatrix} r_2 & \\ r_1 + r_4 & r_2 \end{pmatrix} g^{-1}$  satisfies the second equation in (36). Hence, the possible number of  $x$  is the product of possible numbers of  $B$  as in (37) and  $R$ , which is  $q^2(q - 1)$  and  $q^4$ , respectively. Therefore, we obtain that  $|U_0 \cap \Delta_2(t)| = q^6(q - 1)$  and  $|U_0 \cap \Delta_3(t)| = q^5(q - 1)(q^2 - q - 1)$ .

Now we assume that  $\text{char}(F) \neq 2$ . Similarly, we may assume that

$$x = \left( \begin{array}{c|c} -I_2 - BR & B \\ \hline -2R - RBR & I_2 + RB \end{array} \right) = \left( \begin{array}{c|c} I_2 & \\ \hline R & I_2 \end{array} \right) \left( \begin{array}{c|c} -I_2 & B \\ \hline & I_2 \end{array} \right) \left( \begin{array}{c|c} I_2 & \\ \hline -R & I_2 \end{array} \right),$$

where  $B, R \in \text{M}(2, 2)$ . If  $x \in \Delta_1(t)$ , then  $R = B = 0$ , thus  $|\Delta_1(t) \cap U_0| = 1$ . We assume that at least one of  $R$  and  $B$  is nonzero. We shall find  $|\Delta_2(t) \cap U_0|$ . If  $B = 0$  and  $R \neq 0$  or  $B \neq 0$  and  $R = 0$ , then  $x$  is of the following form

$$(38) \quad x = \left( \begin{array}{c|c} -I_2 & \\ \hline -2R & I_2 \end{array} \right) \text{ or } \left( \begin{array}{c|c} -I_2 & B \\ \hline & I_2 \end{array} \right),$$

respectively. It follows by Proposition 2.2 that they are all in  $\Delta_2(t)$  and the number of possible  $x$  is  $q^4 - 1$  for each case. From now on we assume that  $B \neq 0$  and  $R \neq 0$ .

If  $y \in \Delta_1(t)$  such that  $x \sim y$ , then the involution  $y$  is of the following form

$$y = \left( \begin{array}{c|c} I_2 & \\ \hline R & I_2 \end{array} \right) \left( \begin{array}{c|c} M & \\ \hline & N \end{array} \right) \left( \begin{array}{c|c} I_2 & \\ \hline -R & I_2 \end{array} \right)$$

for some involutions  $M, N$  such that  $RM = NR$  and  $BN = MB$ . Let  $M = gI(2, 1)g^{-1}$  and  $N = hI(2, 1)h^{-1}$  for some  $g, h \in \text{GL}_2(F)$ . Then, we have

$$B = g \begin{pmatrix} a & \\ & b \end{pmatrix} h^{-1} \text{ and } R = h \begin{pmatrix} c & \\ & d \end{pmatrix} g^{-1}$$

for some  $a, b, c, d \in F$ . Hence, if  $x \in \Delta_2(t)$ , then  $x$  is of the following form

$$x = \left( \begin{array}{cc|cc} -1+ac & & a & \\ & -1+bd & & b \\ \hline -2c-ac^2 & & 1+ac & \\ & -2d-bd^2 & & 1+bd \end{array} \right),$$

up to conjugation in the subgroup  $H'$  of block diagonal matrices. Using this, we obtain an  $H'$ -orbit decomposition with the following representatives

$$\left( \begin{array}{cc|cc} -1 & & & \\ & \lambda-1 & & 1 \\ \hline & & 1 & \\ & -\lambda^2-2\lambda & & \lambda+1 \end{array} \right), \left( \begin{array}{cc|cc} -1 & & & \\ & \lambda-1 & & 1 \\ \hline -2 & & 1 & \\ & -\lambda^2-2\lambda & & \lambda+1 \end{array} \right), \left( \begin{array}{cc|cc} \lambda-1 & & 1 & \\ & \lambda-1 & & 1 \\ \hline -\lambda^2-2\lambda & & \lambda+1 & \\ & -\lambda^2-2\lambda & & \lambda+1 \end{array} \right)$$

with  $\lambda \neq 0$  and

$$\left( \begin{array}{cc|cc} -1 & & 1 & \\ & -1 & & \\ \hline & & 1 & \\ & -2 & & 1 \end{array} \right), \left( \begin{array}{cc|cc} -1+\lambda_1 & & 1 & \\ & -1+\lambda_2 & & 1 \\ \hline -\lambda_1(2+\lambda_1) & & 1+\lambda_1 & \\ & -\lambda_2(2+\lambda_2) & & 1+\lambda_2 \end{array} \right)$$

with  $\lambda_1 \neq \lambda_2$ . Then, the total order of all the orbits is

$$q^2(q^2-1)^2 + q^2(q-1)^3(q+1)^2 + q(q^2-1)(q-1)^2 + (q^2-1)^2 + \frac{q^3(q-1)^3(q+1)^2}{2}.$$

Hence, together with case (38) we have

$$|\Delta_2(t) \cap U_0| = \frac{1}{2}(q-1)(q+1)^2(q^5 - q^3 + 2q^2 + 2), |\Delta_3(t) \cap U_0| = \frac{1}{2}q(q-1)^2(q+1)(q^4 + 3q^2 + 2q + 2).$$

□

Combining Lemmas 3.2, 3.3, and 3.4 with Theorem 1.1 immediately yields the following result.

**Proposition 3.5.** *Let  $G = \text{GL}_4(F)$  over a finite field  $F$  with  $q$  elements. Then,  $\text{Diam } \Gamma(G, X_2) = 3$ . In particular,*

$$\begin{aligned} |\Delta_1(t)| &= \begin{cases} 2q^4 - q^3 - 2q^2 + q - 1 & \text{if } \text{char}(F) = 2, \\ q^2(q+1)^2 + 1 & \text{if } \text{char}(F) \neq 2. \end{cases} \\ |\Delta_2(t)| &= \begin{cases} q^2(q^5 + q^4 - q^3 - 4q^2 + q + 2) & \text{if } \text{char}(F) = 2, \\ \frac{1}{2}(q-1)(q+1)(q^6 + 3q^5 + q^4 + 3q^3 + 8q^2 + 4q + 4) & \text{if } \text{char}(F) \neq 2. \end{cases} \\ |\Delta_3(t)| &= \begin{cases} q^4(q^4 - q^3 - q^2 + 1) & \text{if } \text{char}(F) = 2, \\ \frac{1}{2}q(q-1)^2(q+1)(q^4 + 5q^2 + 6q + 4) & \text{if } \text{char}(F) \neq 2. \end{cases} \end{aligned}$$

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